Partial Differential Equations

lecture notes

by

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1 Introduction

About notations and conventions:

- $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}$, $n \geq 2$
- $\Omega \subset \mathbb{R}^n$ open set (*domain*: open and connected)
- $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$
- $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ for all $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$
- $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$, $r > 0$ is an open ball. $\partial B(x, r)$ is the boundary of the ball B and the closure of the ball B is $\overline{B}(x, r) = B \cup \partial B$.
- $u : \Omega \to \mathbb{R}$, $u(x) = u(x_1, \dots, x_n)$.
- $\frac{\partial}{\partial x_i} u(x) = \lim_{\varepsilon \to 0} \frac{u(x + \varepsilon e_i) - u(x)}{\varepsilon}$, if it exists. We denote $\partial_x := \frac{\partial}{\partial x}$.
- $\nabla u(x) = Du(x) = (\partial_{x_1} u(x), \partial_{x_2} u(x), \dots, \partial_{x_n} u(x))$
- $u = u(*, \dots, *)$ denotes a function u dependant only on the variables $*, \dots, *$ given in the parenthesis.
- Hessian matrix: $D^2 u(x) := \begin{pmatrix} \partial_{x_1 x_1} u & \partial_{x_1 x_2} u & \cdots & \partial_{x_1 x_n} \\ \partial_{x_2 x_1} u & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \partial_{x_n x_1} u & \cdots & \cdots & \partial_{x_n x_n} u \end{pmatrix}$
- $(A)_{ij} = a_{ij}$, the item on the ith row and in the jth column in the matrix A.
- $(D^2 u(x))_{ij} = \partial_{x_i} \partial_{x_j} u(x)$
- $\Delta u(x) = \operatorname{trace}(D^2 u(x))$
- $F(x) = (F^1(x), \dots, F^n(x))$, $F^i : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$.

$$\operatorname{div} F = \partial_{x_1} F^1(x) + \partial_{x_2} F^2(x) + \cdots + \partial_{x_n} F^n(x) = \sum_{i=1}^{n} \partial_{x_i} F^i(x).$$

- $\gamma(x) : \mathbb{R}^n \to \mathbb{R}$.

$$\gamma(x) F(x) = \left(\gamma(x) F^1(x), \dots, \gamma(x) F^n(x)\right).$$

- *Support* of a function u is the set: $\operatorname{spt} u := \overline{\{x : u(x) \neq 0\}}$.
- $f : \Omega \to \Omega$. $f(x) = (f^1(x), f^2(x), \dots, f^n(x))$

$$(Df(x))_{ij} = \partial_{x_j} f^i(x).$$

- $(A)_{ij} = a_{ij}$. $(\text{Cof } A)_{ij} = \left(A^{\#}\right) = \frac{\partial}{\partial_{a_{ij}}}(\det A)$.
  $\left\langle A^{\#}, A\right\rangle = \sum_{i,j=1}^{n} \left(A^{\#}\right)_{ij} a_{ij} = n \det A$.
- A^T is the *transpose* of matrix A.
- $⨍_\Omega u(x)\ \mathrm{d}x = \dfrac{1}{|\Omega|} \int_\Omega u(x)\ \mathrm{d}x$, where $|\Omega| = \int_\Omega 1\ \mathrm{d}x$ is the *volume* of Ω, is the *mean value of the function* u in Ω.
- We denote $A \subset\subset B$, if A has compact closure in B. We say A is *strictly contained* in B. We may also denote $\overline{A} \subset B$, when A and B are open.

Example 1.1. 1) Linear transport equation. $u : \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}$, $n \geq 1$, $t \geq 0$.

$$\partial_t u(x,t) + \sum_{i=1}^{n} b_i(x,t) \partial_{x_i} u(x,t) = 0 \tag{1.1}$$

where $b_i = \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}$ are given for all $i = 1, \dots, n$.

2) Laplace equation. $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

$$\Delta u(x) := \partial_{x_1 x_1} u(x) + \partial_{x_2 x_2} u(x) + \cdots + \partial_{x_n x_n} u(x) = 0 \tag{1.2}$$

for all $x \in \Omega$.

3) Heat equation. $u : \Omega \times \mathbb{R}^t \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, $(x,t) \in \Omega \times \mathbb{R}^t$.

$$\partial_t u(x,t) - \underbrace{\left(\partial_{x_1 x_1} u(x,t) + \partial_{x_2 x_2} u(x,t) + \cdots + \partial_{x_n x_n} u(x,t)\right)}_{=:\Delta_x u} = 0 \tag{1.3}$$

for all $x \in \Omega$, $t \geq 0$.

4) Wave equations.

$$\partial_{tt} u(x,t) - \Delta_x u = 0 \tag{1.4}$$

Linear: u, v solutions $\Rightarrow \alpha u + \beta v$ solutions for all $\alpha, \beta \in \mathbb{R}$.

5) Minimal surface equation. $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $x \in \Omega$.

$$\operatorname{div}\left(\frac{\Delta u(x)}{\sqrt{1 + |\nabla u(x)|}}\right) = \sum_{i=1}^{n} \partial_{x_i} \left(\frac{\partial_{x_i} u(x)}{\sqrt{1 + |\nabla u(x)|^2}}\right) = 0. \tag{1.5}$$

Case $n = 2$. $u : \mathbb{R}^2 \to \mathbb{R}$. Given $B(0,1) =: \Omega \subset \mathbb{R}^2$. Want to find a function $u : \Omega \to \mathbb{R}$ such that the graph of u, $\Gamma(u)$, has the minimal surface area and $u(x) = u_0(x)$ for all $x \in \partial\Omega$. The surface area of $\Gamma(u)$:

$$A(u) := \int_\Omega \sqrt{1 + |\nabla u(x)|^2}.$$

Suppose that u is the function such that $u = u_0$ on $\partial\Omega$ and $\Gamma(u)$ has the minimal surface area.
Claim: u is a solution to (1.5).
Proof. For all $\varphi \in C_0^\infty(\Omega)$, $t \in \mathbb{R}$, $\operatorname{spt}\varphi \subset \Omega$, let

$$v(x) := u(x) + t\varphi.$$

Then $v(x) = u_0(x)$ for all $x \in \partial\Omega$. Let

$$h(x) := A(u(x) + t\varphi(x)).$$

Then

$$A(u) \leq A(h) = A(u + t\varphi).$$

Define $h(t) = A(u + t\varphi)$. Function h reaches its minimum at $t = 0$. Therefore $h'_{t=0}(t) = 0$.

$$\begin{aligned} h(t) &= A(u + t\varphi) = \int_\Omega \sqrt{1 + |\nabla u(x) + t\nabla\varphi(x)|^2}\ \mathrm{d}x \\ h'(t)|_{t=0} &= \int_\Omega \left.\frac{(\nabla u(x) + t\nabla\varphi(x)) \cdot \nabla\varphi(x)}{\sqrt{1 + |\nabla u(x) + t\nabla\varphi(x)|^2}}\right|_{t=0} \mathrm{d}x \\ &= \int_\Omega \frac{\nabla u(x) \cdot \nabla\varphi(x)}{\sqrt{1 + |\nabla u(x)|^2}}\ \mathrm{d}x \\ &\overset{\text{i.b.p.}}{=} -\int_\Omega \operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}\right)\varphi(x)\ \mathrm{d}x = 0 \end{aligned}$$

for all $\varphi \in C_0^\infty(\Omega)$. Here i.b.p. denotes inegration by part, see Theorem 1.11. Therefore[1] $\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}}\right) = 0.$

6) Poisson equation.

$$\Delta u(x) := \partial_{x_1x_1}u(x) + \partial_{x_2x_2} + \cdots \partial_{x_nx_n}u(x) = f(x) \tag{1.6}$$

where $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is given and $u(x) : \Omega \to \mathbb{R}$ is unknown.

7) Nonlinear Poisson equation.

$$\Delta u = g(u) \tag{1.7}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given, for example: $g(t) = -t + t^3$: $\Delta u = -u + u^3$

[1] Exercise. Let $\Omega \subset \mathbb{R}^n$ and $u \in C(\Omega)$. Suppose that $\int_\Omega u\varphi\ \mathrm{d}x = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Show that $u(x) = 0$ for all $x \in \Omega$.

8) Helmholtz's equation. (Eigenvalue problem for Δ.)

$$-\Delta u = \lambda u \tag{1.8}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ unknown and $\lambda \in \mathbb{R}$.

9) Biharmonic equation.

$$\Delta^2 u = \Delta(\Delta u) := \sum_{i=1}^{n} \partial_{x_i x_i} \left(\sum_{j=1}^{n} \partial_{x_j x_j} u(x) \right) = 0. \tag{1.9}$$

Note: Solution to Laplace equation is a solution of Biharmonic equation.

10) Eikonal equation.

$$|\nabla u(x)| = \sqrt{|\partial_{x_1} u(x)|^2 + \cdots + |\partial_{x_n} u(x)|^2} = 1 \tag{1.10}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$.

11) Monge-Ampère equation.

$$\det\left(D^2 u(x)\right) = f(x) \tag{1.11}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ unknown, $f : \Omega \to \mathbb{R}$.

12) Hamilton-Jacobi equation.

$$\partial_t u + H(\nabla u) = f \tag{1.12}$$

where $u = u(x,t)$, $f = f(x,t)$, $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. $H : \mathbb{R}^n \to \mathbb{R}$ given and $f : \Omega \times \mathbb{R}^t \to \mathbb{R}$. For example: $H(y) = |y|^2$: $\partial_t u + |\nabla u|^2 = f$.

13) Euler's equation for incompresible ideal fluids. $\Omega \subset \mathbb{R}^n$, $n = 2, 3$. $u : \Omega \times \mathbb{R}^t \to \mathbb{R}^n$, $u(x,t)$ denotes the velocity of the fluid at the position x at given time t. $u(x,0) = u_0(x)$ is given.

$$\begin{cases} \partial_t u + Du \cdot u & = & -\nabla P & \\ \operatorname{div} u & = & 0 & \\ u(x,0) = u_0(x), & & \operatorname{div} u_0 = 0 & \text{(initial condition)} \\ \langle u, \nu \rangle & = & 0 \text{ on } \partial\Omega & \text{(boundary condition)} \end{cases} \tag{1.13}$$

where $u(x,t)$ is the velocity and $P(x,t)$ is the pressure.

Let $f(x,t)$ be the place of particle x at given time t such that $f(x,0) = x$. $f : \Omega \to \Omega$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x,t) = u(f(x,t),t), \tag{1.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t^2} f(x,t) = a\left(f(x,t),t\right) = -\nabla P\left(f(x,t),t\right). \tag{1.15}$$

Equation (1.14) implies

$$D\left(\frac{\mathrm{d}}{\mathrm{d}t^2} f(x,t)\right) = \frac{\mathrm{d}}{\mathrm{d}t^2} D(f(x,t)) = D_x u(f(x,t),t) = Du(f(x,t),t) Df(x,t).$$

Thus,

$$\begin{aligned}
\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\left(Df(x,t)\right), \left(Df(x,t)\right)^{\#} \right\rangle &= \left\langle Du(f(x,t),t)Df(x,t), (Df(x,t))^{\#} \right\rangle \\
&= \left\langle Du(f(x,t),t), (Df(x,t))^{\#}\, Df(x,t)^T \right\rangle \\
&= \langle Du(f(x,t),t), \det\left(Df(x,t)\right) I \rangle \\
&= n \det\left(Df(x,t)\right) \operatorname{trace} \underbrace{\left(Du(f(x,t),t)\right)}_{=\operatorname{div} u} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \det \underbrace{\left(D(f(x,t)\right)}_{=:A} \\
&= \sum_{i,j=1}^{n} \frac{\partial}{\partial_{a_{ij}}} \det(A) \frac{\mathrm{d}}{\mathrm{d}t} a_{ij} \\
&= \left\langle (Df(x,t))^{\#}, \frac{\mathrm{d}}{\mathrm{d}t}(Df(x,t)) \right\rangle
\end{aligned}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\det\left(Df(x,t)\right)\right) = \frac{1}{n} \underbrace{\left(\operatorname{div} u(f(x,t),t)\right)}_{=0} \det\left(Df(x,t)\right).$$

which is equivalent with

$$\det\left(Df(x,t)\right) = \det\left(Df(x,0)\right) = 1$$

since the fluid is incompressible and $f(x,0) = x$.

Equation (1.15) implies

$$\partial_t u(f(xt),t) + Du(f(x,t),t) \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)}_{=u(f(x,t),t)} = -\Delta P(f(x,t),t)$$

Denote $f(x,t) = y$. Then for all $y \in \Omega$,

$$\partial_t u(y,t) + Du(y,t) \cdot u(y,t) = -\Delta P(y,t).$$

14) Navier-Stokes equation for incompressible, viscous fluids.

$$\begin{cases} \partial_t u - \Delta u + Du \cdot u &= -\nabla P \\ \operatorname{div} u &= 0 \\ u(x,0) = u_0(x), & \operatorname{div} u_0 = 0 \\ u(x,t) &= 0 \text{ for all } t \geq 0, x \in \partial\Omega. \end{cases} \tag{1.16}$$

where $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$ and $P : \Omega \times \mathbb{R}^+ \to \mathbb{R}$.

Navier-Stokes existence and smoothness problem ($ 1 million problem):
Prove or give a counter example of the following statement ($n = 3$): Given smooth $u_0(x)$, there exists (u, P) smooth in $\Omega \times \mathbb{R}^+$ that solves the equation (1.16) with finite energy

$$\int_\Omega |u(x,t)|^2 \, \mathrm{d}t \leq M < \infty \text{ for all } t > 0.$$

Historic notes:

- Leray (30'), Hopf (50') existence of *weak* solutions.
- Ladyzhenska (50'), $n = 2$.
- Caffarelli-Kuhn-Nirenberg (80').

Fun part:

$$\begin{aligned}\int_\Omega \partial_{x_j} u^i \cdot u^j \cdot u^i \, \mathrm{d}x &= \int_\Omega u^j \partial \left(\frac{1}{2}|u|^2\right) \mathrm{d}x \\ &\overset{\text{i.b.p}}{=} -\int_\Omega \underbrace{\left(\sum_j \partial_{x_j} u^j\right)}_{=\operatorname{div} u = 0} \cdot \frac{1}{2}|u|^2 \, \mathrm{d}x = 0.\end{aligned}$$

Where integration by part (i.p.b) is used in a manner:

$$-\int_\Omega \nabla P \cdot u \, \mathrm{d}x = \int_\Omega P(\underbrace{\operatorname{div} u}_{=0}) \, \mathrm{d}x = 0$$

Definition 1.2 (Partial differential equation)**.** A *partial differential equation* (PDE) is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Definition 1.3 (PDE of order k, multi index α, D^α)**.** Fix $k \geq 1$, an integer. An expression of the form

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0 \tag{1.17}$$

is called a *k-th order partial differential equation*, where

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \cdots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$$

is given and $u : \Omega \to \mathbb{R}$ is the unknown function. We use following notation. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_i \geq 0$ non-negative integer. Set

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

and

$$D^\alpha u(x) = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}} u(x).$$

Remark 1.4. Note that $D^2u(x)$ is used to denote the Hessian matrix of $u(x)$. See the context!

Definition 1.5 ($C(\Omega)$, $C^k(\Omega)$, $C_0^k(\Omega)$)**.** We denote

$$C(\Omega) = \{f : \Omega \to \mathbb{R} : f \text{ is continuous in } \Omega \subset \mathbb{R}^n\},$$
$$C^k(\Omega) = \{f : \Omega \to \mathbb{R} : f \in C(\Omega),\ D^\alpha f \in C(\Omega),\ |\alpha| \leq k\}$$

and

$$C_0^k(\Omega) = \left\{u \in C^k(\Omega) : \operatorname{spt} u = \overline{\{x \in \Omega : u(x) \neq 0\}} \subset \Omega,\ \operatorname{spt} u \text{ compact}\right\}$$

Definition 1.6 (Linear, semiliniear, quasilinear and fully non-linear PDEs)**.**

(i) The PDE (1.17) is called *linear* if it has form

$$\sum_{|\alpha|\leq k} b_\alpha(x) D^\alpha u(x) = f(x),$$

where b_α and f are given.

(ii) The PDE (1.17) is called *semilinear* if it has the form

$$\sum_{|\alpha|=k} b_\alpha(x) D^\alpha u(x) + B(D^{k-1}u, \ldots, Du, u, x) = 0$$

where b_α and B are given.

(iii) The PDE (1.17) is called *quasilinear* if it has the form

$$\sum_{|\alpha|=k} b_\alpha(D^{k-1}u \ldots, Du, u, x) D^\alpha u(x) + B(D^{k-1}u, \ldots, Du, u, x) = 0.$$

where b_α and B are given.

(iv) The PDE (1.17) is called *fully non-linear* if it depends non-linearly upon the highest order derivatives.

Definition 1.7 (Solution to a PDE)**.** $u \in C^k(\Omega)$ is called a *solution* (classical solution) of equation (1.17), if the equation holds for every $x \in \Omega$.

Definition 1.8 (C^1 set, outward unit, outward normal derivative)**.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Ω is called C^1 *set* or *smooth set* and denoted $\Omega \in C^1$, if for all $x_0 \in \Omega$ there exists $r > 0$ and C^1 function $g : \mathbb{R}^{n-1} \to \mathbb{R}$ such that (relabeling the coordinates, if necessary) we have

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_n > g(x_1, \ldots, x_n - 1)\}.$$

If $\Omega \in C^1$, then for all $x_0 \in \partial\Omega$ there exists unique *outward unit*

$$\nu(x_0) = \left(\nu^i(x_0), \nu^2(x_0), \ldots, \nu^n(x_0)\right),\ |\nu(x_0)| = 1.$$

Outward normal derivative is then defined by setting

$$\frac{\partial u}{\partial \nu}(x_0) = \nabla u(x_0) \cdot \nu(x_0) = \sum_{i=1}^n \partial_{x_i} u(x_0) \cdot \nu^i(x_0).$$

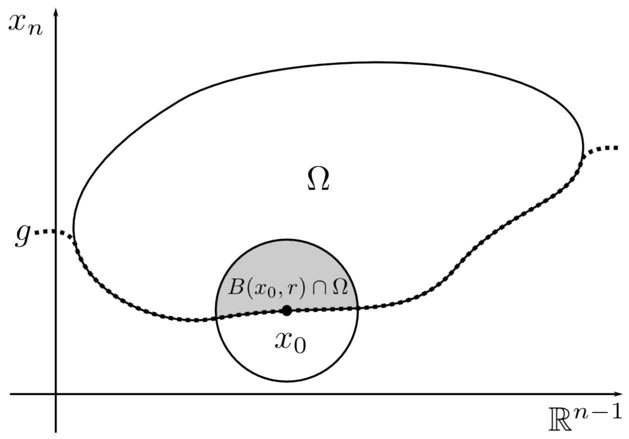

Figure 1: C^1 set

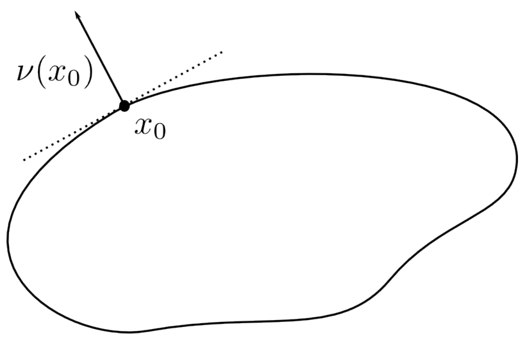

Figure 2: C^1 set and its outward unit at point x_0

Example 1.9.

- For a ball $B(0, r)$ the outward unit is $\nu(x_0) = \frac{x_0}{|x_0|} = \frac{x_0}{r}$.
- **Important!** For a ball $B(y_0, r)$ the outward unit is $\nu(x_0) = \frac{x_0 - y_0}{r}$.

Definition 1.10 ($C^1(\overline{\Omega})$)**.** We denote

$C^1(\overline{\Omega}) = \left\{ u : \Omega \to \mathbb{R} : u \in C^1(\Omega),\ \partial_{x_i} u \text{ has continuous extension to } \overline{\Omega}\ \forall 1 \leq i \leq n \right\}.$

Theorem 1.11 (Integration By Part)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and $u, v \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \partial_{x_i} u \cdot v \ dx = \int_{\partial\Omega} u \cdot v \cdot \nu^i \quad dS(x) - \int_\Omega u \cdot \partial_{x_i} v \ dx,$$

where $\nu(x) = (\nu^1(x), \ldots, \nu^n(x))$ is the outward unit at $x \in \Omega$ and $dS(x)$ indicates the $(n-1)$-dimensional area element at $x \in \partial\Omega$.

Theorem 1.12 (Gauss-Green Divergence Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain and $u \in C^1(\overline{\Omega})$. Then*

$$\int_\Omega \partial_{x_i} u \ dx = \int_{\partial\Omega} u \cdot \nu^i \quad dS(x) \ (i = 1, 2, \ldots, n).$$

Proof. Proof follows by applying integration by part with $v \equiv 1$. $\square$

Corollary 1.13. *Let $F(x) = (F^1(x), F^2(x), \dots, F^n(x)) \in C^1(\Omega) \cap C^1(\overline{\Omega})$. Then*

$$\int_\Omega \operatorname{div} F(x)\ dx = \int_{\partial\Omega} F(x) \cdot \nu(x)\ dS(x).$$

Proof.

$$\int_\Omega \sum_{i=1}^n \partial_{x_i} F^i(x)\ \mathrm{d}x \overset{\text{G-G}}{=} \int_{\partial\Omega} \sum_{i=1}^n F^i(x) \cdot \nu^i(x)\ \mathrm{d}S(x)$$

$\square$

Note: Because the writer is lazy, we don't necessarily denote every "$\cdot$" anymore.

Theorem 1.14 (Green's formulae)**.** *Let $u, w \in C^2(\Omega) \cap C^1(\overline{\Omega})$. Then*

(i)

$$\int_\Omega \Delta u(x)\ dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x)\ dS(x).$$

(ii)

$$\int_\Omega \nabla u(x) \cdot \nabla w(x)\ dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x) \cdot w(x)\ dS(x) - \int_\Omega (\Delta u(x)) \cdot w(x)\ dx.$$

(iii)

$$\int_\Omega u(x) \cdot \Delta w(x) - w(x) \cdot \Delta u(x)\ dx = \int_{\partial\Omega} u(x) \cdot \frac{\partial u}{\partial \nu}(x) - w(x) \cdot \frac{\partial u}{\partial \nu}(x)\ dS(x).$$

Proof. Remember the equation

$$\int_\Omega \partial_{x_i} f(x) \cdot g(x)\ \mathrm{d}x = \int_{\partial\Omega} f(x) \cdot g(x) \cdot \nu^i(x)\ \mathrm{d}S(x) - \int_\Omega f(x) \cdot \partial_{x_i} g(x)\ \mathrm{d}x. \quad (1.18)$$

Proof of part (i). Employ (1.18) with $f = \partial_{x_i} u$ and $g \equiv 1$ or apply Corollary 1.13 to $F(x) = \nabla u(x)$ to obtain

$$\int_\Omega \Delta u(x) = \int_\Omega \operatorname{div}(\nabla u(x)) = \int_{\partial\Omega} \nabla u(x) \cdot \nu(x)\ \mathrm{d}S(x).$$

$\square_{\text{(i)}}$

Proof of part (ii). Employ (1.18) with $f = w$ and $g = \partial_{x_i} u$. $\square_{\text{(ii)}}$

Proof of part (iii). Write (ii) with u and w interchanged and then substract. $\square_{\text{(iii)}}$

$\square$

2 First order linear equations

2.1 Simple PDE

Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u = u(x, y)$.

$$\partial_x u(x, y) = 0.$$

Solution: $u(x, y) = g(y)$ for a function $g : \mathbb{R} \to \mathbb{R}$ such that $g \in C^1(\mathbb{R})$. Is this the only solution?

Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u = (x, y)$.

$$a\partial_x u(x, y) + b\partial_y u(x, y) = 0. \tag{2.1}$$

Solution: $u(x, y) = g(-bx + ay)$ where $g \in C^1(\mathbb{R})$.

Check:

$$\begin{aligned} a\partial_x u(x, y) &= g'(-bx + ay) \cdot (-b) \cdot a \\ b\partial_y u(x, y) &= g'(-bx + ay) \cdot a \cdot b. \end{aligned}$$

Also, is this the only solution? Let's see how this solution can be deduced.

2.2 Method of characterictic curves

Let us assume that we are given a solution u to (2.1). Fix $(x_0, y_0) \in \mathbb{R}^2$ to find a curve $\Gamma \subset \mathbb{R}^2$

$$\Gamma = \{(x(s), y(s)) : x(0) = x_0, y(0) = y_0, s \in \mathbb{R}\}.$$

Let

$$z(s) := u((x(s), y(s)).$$

Then

$$\frac{\partial z}{\partial s}(s) = \partial_x u(x(s), y(s)) \cdot \frac{\partial}{\partial s} x(s) + \partial_y u(x(s), y(s)) \cdot \frac{\partial}{\partial s} y(s). \tag{2.2}$$

Now letting

$$\begin{cases} \frac{\partial}{\partial s} x(s) &=: a \\ x(0) &= x_0 \end{cases} \text{and} \begin{cases} \frac{\partial}{\partial s} y(s) &=: b \\ y(0) &= y_0 \end{cases}$$

imply

$$\begin{cases} x(s) &= as + x_0 \\ y(s) &= bs + y_0 \end{cases}$$

and we obtain

$$bx(s) - ay(s) = bx_0 - ay_0. \tag{2.3}$$

Thus,

$$\frac{\partial}{\partial s} z(s) = a\partial_x u(x, y) + b\partial_y u(x, y) = 0.$$

Therefore $z(s) = z(0)$ for all $s \in \mathbb{R}$ so $u(x(s), y(s)) = u(x_0, y_0)$. Equation (2.3) then implies that for all $(x, y) \in \mathbb{R}^2$, $u(x, y) = g(bx - ay)$ is a solution to (2.1) for any $g \in C^1(\mathbb{R})$.

2.2.1 Adding boundary condition

Let $u : \mathbb{R}^2 \to \mathbb{R}$, $u = (x, y)$. Now add boudary condition to equation (2.1).

$$\begin{cases} a\partial_x u(x,y) + b\partial_y u(x,y) &= 0 \\ \qquad u(0,y) &= y^2 \quad \text{(boundary condition)} \end{cases} \tag{2.4}$$

Fix $(x_0, y_0) \in \mathbb{R}^2$ to find a curve $\Gamma \subset \mathbb{R}^2$

$$\Gamma = \{(x(s), y(s)) : x(0) = x_0, y(0) = y_0, s \in \mathbb{R}\}.$$

Let

$$z(s) := u((x(s), y(s)).$$

Now letting

$$\begin{cases} \frac{\partial}{\partial s}x(s) &=: a \\ x(0) &= x_0 \end{cases} \text{and} \begin{cases} \frac{\partial}{\partial s}y(s) &=: b \\ y(0) &= y_0 \end{cases}$$

implies

$$\begin{cases} x(s) &= as + x_0 \\ y(s) &= bs + y_0 \end{cases}.$$

Thus,

$$\frac{\partial}{\partial s}z(s) = a\partial_x u(x,y) + b\partial_y u(x,y) = 0.$$

Therefore $z(s) = z(0)$ for all $s \in \mathbb{R}$ so

$$u(x(s), y(s)) = u(x_0, y_0). \tag{2.5}$$

Choose $s_0 = \frac{-x_0}{a}$. Then $x(s_0) = 0$, $y(s_0) = -\frac{b}{a}x_0 + y_0$ and

$$\begin{aligned} z(s_0) &= u(\underbrace{x(s_0)}_{=0}, \underbrace{y(s_0)}_{-\frac{b}{a}x_0+y_0}) \\ &\overset{\substack{\text{boudary}\\ \text{condition}}}{=} \left(-\frac{b}{a}x_0 + y_0\right)^2. \end{aligned}$$

From equation (2.5) it follows that

$$u(x,y) = \left(-\frac{b}{a}x + y\right)^2.$$

Theorem 2.1. *Problem (2.4) has exactly one solution.*

Proof. Existence. $u(x,y) = \left(-\frac{b}{a}x + y\right)^2$ is a solution. $\square_{\text{Existence}}$

Uniqueness. Let v be a another solution. Then $w = u - v$ is a solution to

$$\begin{cases} a\partial_x u(x,y) + b\partial_y u(x,y) &= 0 \\ \qquad u(0,y) &= 0 \end{cases}.$$

Calculate and obtain $w \equiv 0$ on $\mathbb{R}^2$. $\square_{\text{Uniqueness}}$

$\square$

Remark 2.2. Solution is similar to any boundary condition $u(0,y) = g(y)$.

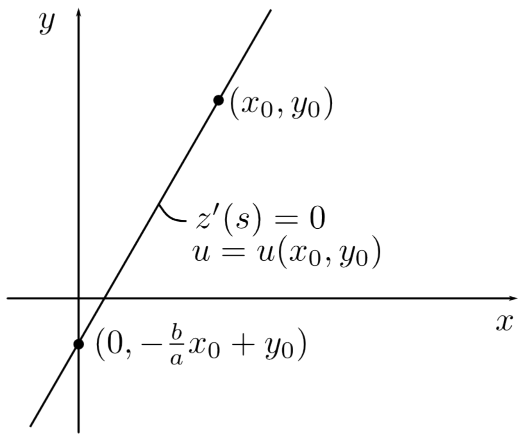

Figure 3: Characteristic curve of u

2.3 Another generalization

$$\begin{cases} -y\partial_x u(x,y) + x\partial_y u(x,y) &= 0 \\ u(0,y) &= y^2 \end{cases} \tag{2.6}$$

Now similar calculation follows. Set

$$\frac{\partial}{\partial s}x(s) = -y(s) \text{ and } \frac{\partial}{\partial s}y(s) = x(s).$$

This leads to the equations

$$\begin{cases} \frac{\partial}{\partial s}x(s) &= -y(s) \\ x(0) &= x_0 \end{cases} \text{ and } \begin{cases} \frac{\partial}{\partial s}y(s) &= x(s) \\ y(0) &= y_0 \end{cases}.$$

which imply

$$\frac{\partial^2}{\partial s^2}x(s) = -\frac{\partial}{\partial s}y(s) = -x(s).$$

Thus

$$\begin{cases} x(s) &= \sqrt{x_0^2 + y_0^2} \cdot \cos(s+\alpha) \\ y(s) &= \sqrt{x_0^2 + y_0^2} \cdot \sin(s+\alpha) \end{cases}.$$

where $\cos\alpha = \dfrac{x_0}{\sqrt{x_0^2 + y_0^2}}$.

Choose s_0 such that $x(s_0) = 0$ and solve $u(x,y)$ for arbitrary $(x,y) \in \mathbb{R}^2$. Then try to mimic the uniqueness theorem for the previous equation.

Also, it can be shown that all solutions are of the form $u(x,y) = g(x^2 + y^2)$ for all $(x,y) \in \mathbb{R}^2$ and for some $g : \mathbb{R} \to \mathbb{R}$.

Remark 2.3. Note that our boundary condition works and others do not, for example $u(0,y) = y^3$.

Try the same algorithm for problems

$$\begin{cases} -y\partial_x u(x,y) + x\partial_y u(x,y) &= u(x,y),\ f(x,y),\ f(u,x,y)\dots \\ u(0,y) &= g(y) \end{cases}.$$

Details will be covered in the exercises.

Example 2.4. Solve equation

$$\begin{cases} \partial_x u(x,y) + \partial_y u(x,y) & = & x \\ u(0,y) & = & y \end{cases}. \tag{2.7}$$

Solution: Fix (x_0, y_0). Suppose u is a solution. Consider

$$\Gamma := \{(t + x_0, t + y_0) : t \in \mathbb{R}\}.$$

Define

$$z(t) := u(t + x_0, t + y_0).$$

Then $z'(t) = t + x_0$ by the first equation in (2.7). By The Fundamental Theorem of Calculus we have

$$z(t) = z(0) + \int_0^t z'(s) \, \mathrm{d}s = u(x_0, y_0) + \int_0^t s + x_0 \, \mathrm{d}s = u(x_0, y_0) + \frac{1}{2}t^2 + x_0 t.$$

Choosing $t = -x_0$ we get

$$u(x_0, y_0) + \frac{x_0^2}{2} - x_0^2 = z(-x_0) = u(0, y_0 - x_0) \overset{\substack{\text{boundary} \\ \text{condition}}}{=} y_0 - x_0$$

which implies

$$u(x_0, y_0) = \frac{x_0^2}{2} - x_0 + y_0.$$

Since (x_0, y_0) is arbitrary, we have $u(x, y) = \frac{x^2}{2} - x + y$ for all $(x, y) \in \mathbb{R}^2$.

3 Linear transport equation

Let $u : \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}$, $u = u(x, t)$, $x \in \mathbb{R}^n$, $t \geq 0$.

$$\begin{cases} \partial_t u(x,t) + b \cdot \nabla_x u(x,t) & = & 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^t \\ u(x,0) & = & g(x) & \text{for all } x \in \mathbb{R}^n, \end{cases}. \tag{3.1}$$

where $b = (b^1, b^2, \dots b^n) \in \mathbb{R}^n$ and $\nabla_x u(x, t) \in \mathbb{R}^n$ does not include time t.

Solution: Fix $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^t$ to find curve

$$\Gamma := \{x(s) \in \mathbb{R}^n : x(0) = x_0, \ s \in \mathbb{R}\}$$

where $x(s) = (x^1(s), x^2(s), \dots, x^n(s))$.

Define

$$z(s) := u(x(s), t_0 + s).$$

Then for $s = 0$ we have $z(0) = u(x_0, t_0)$ and

$$\frac{\partial}{\partial s} z(s) = \partial_t u(x(s), t_0 + s) + \nabla_x u(x(s), t_0 + s) \cdot \frac{\partial}{\partial s} x(s). \tag{3.2}$$

If $\frac{\partial}{\partial s}x(s) = b$, equation (3.2) equals to 0 and we have $z(s) = z(0) = u(x_0, t_0)$ for all $s \in \mathbb{R}$.

Now,

$$\begin{cases} \frac{\partial x(s)}{\partial s} & = & b \\ x(0) & = & x_0 \end{cases}$$

implies $x(s) = x_0 + bs$. So

$$z(s) = u(x_0 + bs, t_0 + s).$$

Setting s to $-t_0$ gives $z(s) = z(-t_0) = u(x_0 - t_0 b, 0) = g(x_0 - t_0 b)$. Thus

$$u(x, t) = g(x - tb).$$

Remark 3.1. Similar method can be used for the problem

$$\begin{cases} \partial_t u(x,t) + b(x) \cdot \nabla_x u(x,t) & = & 0 & \text{in } \mathbb{R}^n \times \mathbb{R}^t \\ u(x,0) & = & g(x) & \text{for all } x \in \mathbb{R}^n, \end{cases} \tag{3.3}$$

or the problem

$$\begin{cases} \partial_t u(x,t) + b \cdot \nabla_x u(x,t) & = & f(x,t) & \text{in } \mathbb{R}^n \times \mathbb{R}^t \\ u(x,0) & = & g(x) & \text{for all } x \in \mathbb{R}^n, \end{cases} \tag{3.4}$$

where f is given *cost function*.

Note that now

$$\begin{aligned} \int_{-t_0}^{0} f(x_0 + bs, t_0 + s)\ \mathrm{d}s &= \int_{-t_0}^{0} \frac{\partial}{\partial s} z(s)\ \mathrm{d}s = z(0) - z(-t_0) \\ &= u(x_0, t_0) - u(x_0 - bt_0, 0). \end{aligned}$$

Thus

$$u(x,t) = g(x - bt) + \int_{-t_0}^{0} f(x + bs - t + s)\ \mathrm{d}s.$$

Remark 3.2. Problems with the coefficient b depending on u (or even ∇u) are in general very difficult!

Example 3.3. 1) Burgers equation. $u : \mathbb{R} \times \mathbb{R}^t \to \mathbb{R}$, $u = u(x,t)$.

$$\partial_t u(x,t) + \underbrace{u(x,t)}_{=b?} \partial_x u(x,t) = 0$$

2) Hamilton-Jacobi equation. $u : \mathbb{R}^n \times \mathbb{R}^t \to \mathbb{R}$, $u = u(x,t)$.

$$\partial_t u + \underbrace{|\nabla u|^2}_{=\underbrace{\nabla u}_{=b?} \cdot \nabla u} = 0.$$

3) 2D-Euler equation. $u : \mathbb{R}^2 \times \mathbb{R}^t \to \mathbb{R}^2$, $u = u(x,t) = u(u^1(x,t), u^2(x,t))$.

$$\begin{cases} \partial_t u + Du \cdot u = -\nabla p \\ \operatorname{div} u = 0 \end{cases},$$

where $p : \mathbb{R}^2 \to \mathbb{R}$, $p \in C^2(\mathbb{R}^2)$.

The first equatio is equal to

$$\begin{pmatrix} \partial_t u^1 \\ \partial_t u^2 \end{pmatrix} + \begin{pmatrix} \partial_{x_1} u^1 & \partial_{x_2} u^1 \\ \partial_{x_1} u^2 & \partial_{x_2} u^2 \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = - \begin{pmatrix} \partial_{x_1} p \\ \partial_{x_2} p \end{pmatrix} \tag{3.5}$$

From the equation (3.5) we obtain

$$\partial_t u^1 + \sum_{j=1}^{2} \partial_{x_j} u^1 \cdot u^j = -\partial_{x_1} p \tag{3.6}$$

$$\partial_t u^2 + \sum_{j=1}^{2} \partial_{x_j} u^2 \cdot u^j = -\partial_{x_2} p. \tag{3.7}$$

Applying ∂_{x_2} to both sides of equation (3.6) and ∂_{x_1} to both sides of equation (3.7) we obtain

$$\partial_{x_2}\partial_t u^1 + \sum_{j=1}^{2} \partial_{x_2}\partial_{x_j} u^1 \cdot u^j = -\partial_{x_2}\partial_{x_1} p = -\partial_{x_1}\partial_{x_2} p = \partial_{x_1}\partial_t u^2 + \sum_{j=1}^{2} \partial_{x_1}\partial_{x_j} u^2 \cdot u^j.$$

This implies

$$\partial_t \underbrace{\left(\partial_{x_2} u^1 - \partial_{x_1} u^2\right)}_{=:\omega,\ \text{vorticity}} = 0.$$

So we have an equation

$$\partial_t \omega + \underbrace{u}_{=:b?} \cdot \nabla \omega = 0.$$

4 Laplace equation

Remember the Laplace (Laplacian) equation:

$$0 = \Delta u(x) = \sum_{i=1}^{n} \partial_{x_i x_i} u(x) = \operatorname{div}\left(\nabla u(x)\right). \tag{4.1}$$

Definition 4.1 (Harmonic function)**.** We say that $u \in C^2(\Omega)$ is a *harmonic function* in $\Omega \subset \mathbb{R}^n$ if $\Delta u(x) = 0$ for all $x \in \Omega$.

Example 4.2. Some harmonic functions:

1. $u \equiv c$

2. $u(x) = x_1^2 - x_2^2$, $u(x) = x_1 x_2$

3. $u(x) = b \cdot x + c = \sum_{i=1}^n b_i x_i + c$

4. $u(x) e^{x_1} \sin x_2$. Check: $\partial_{x_1 x_1} u = e^{x_1} \sin x_2$, $\partial_{x_2 x_2 1} u = e^{x_1} \sin x_2$.

5. $u(r, \theta) = r^k \sin(k\theta) = \text{Re}(z^k)$, $u(r, \theta) = r^k \cos(k\theta) = \text{Im}(z^k)$, when $z = \left(e^{i\theta} |z|\right)^k$.

Example 4.3 (Derivation of equation)**.** Let $n = 3$. Let $F : \Omega \to \mathbb{R}^3$, $F(x) = (F^1(x), F^2(x), F^3(x))$ be a an electric field. Let $x, y \in \Omega$. Pick $\gamma : [a, b] \to \mathbb{R}^3$ such that $\gamma(a) = x$ and $\gamma(b) = y$. Then the integral

$$\int_x^y F \cdot \mathrm{d}s = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \; \mathrm{d}t$$

does not depend on the choice of γ.

Fix $x_0 \in \Omega$. Then

$$u(x) = \int_{x_0}^x F \cdot \; \mathrm{d}s$$

implies $F(x) = \nabla u(x)$, if $u(x_0) = 0$.

$$\begin{aligned} \int_{x_0}^x F \cdot \; \mathrm{d}s &= \int_a^b \underbrace{\nabla u(\gamma(t) \cdot \gamma'(t))}_{= \frac{\mathrm{d}}{\mathrm{d}t} u(\gamma(t))} \; \mathrm{d}t \\ &= u(\gamma(b)) - u(\gamma(a)) = u(x) - u(x_0). \end{aligned}$$

Fact: Electric field is divergence free, that is $\operatorname{div} F = 0$. Thus

$$\Delta u = \operatorname{div}(\nabla u) = 0.$$

The energy of electric field is

$$\begin{aligned} E &= \int_\Omega |F(x)|^2 \; \mathrm{d}x \\ &= \int_\Omega |\operatorname{div} u(x)|^2 \; \mathrm{d}x. \end{aligned}$$

Theorem 4.4 (Fundamental solutions)**.** *$\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,*

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi} \log |x| , & n = 2 \\[2ex] \dfrac{1}{n(n-2)\alpha_n} |x|^{2-n} , & n \geq 3 \end{cases} \tag{4.2}$$

defined on $\mathbb{R}^n \setminus \{0\}$ is the fundamental solution *of Laplace equation. Here $\alpha_n = |B(0, 1)|$ is the* volume of the unit ball *of $\mathbb{R}^n$.*

Proof. $n \geq 3$: $\Delta\Phi(x) = 0$ for all $x \neq 0$. (Exercise.[2])

$n = 2$: Oh, you just calculate! □

Remark 4.5.
- For all $y \in \mathbb{R}^n$, $\Phi(x-y) = \frac{1}{n(n-2)\alpha_n} |x-y|^{2-n}$ is harmonic, if $x \neq y$.
- For all $f \in C_0^\infty(\mathbb{R}^n)$ define

$$u(x) = \int_{\mathbb{R}}^{n} \Phi(x-y)f(y)\mathrm{d}y = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \Phi(x-y)f(y)\mathrm{d}y.$$

The limit exists: Let $(\varepsilon_i)_{i=0}^\infty$ be such that $\varepsilon_i \to 0$. Do the change of variables[3]

$$\begin{aligned}
\left| \int_{\mathbb{R}^n \setminus B(x,\varepsilon_i)} \Phi(x-y)f(y)\mathrm{d}y \right| &\overset{x-y=z}{=} \left| \int_{\mathbb{R}^n \setminus B(0,\varepsilon_i)} \Phi(z)f(x-z)\mathrm{d}z \right| \\
&\leq \int_{\mathbb{R}^n \setminus B(0,\varepsilon_i)} |\Phi(z)| \, |f(x-z)| \, \mathrm{d}z \\
&= \int_{B(0,R+|x|\setminus B(0,\varepsilon_i)} |\Phi(z)| \underbrace{|f(x-z)|}_{\leq M} \mathrm{d}z \\
&= M \int_{B(0,R+|x|\setminus B(0,\varepsilon_i)} \underbrace{|\Phi(z)|}_{=\frac{c_n}{|z|^{n-2}}} \mathrm{d}z \\
&\leq C_n'
\end{aligned}$$

since $f \in C_0^\infty(\mathbb{R}^n)$ and it has a compact support in a ball $B(0, R+|x|)$ and therefore $|f|$ is bounded by M. It then suffces to prove that

$$(A_i)_{i=0}^\infty := \left(\int_{\mathbb{R}^n \setminus B(x,\varepsilon_i)} \Phi(x-y)f(y)\mathrm{d}y \right)_{i=0}^\infty$$

is Cauchy sequence by using the previous approximation.

---

[2]Calculate $\Delta |x|^\alpha$.

[3]"I don't know why you guys do the change of variables. It's lot more difficult that way." — Zhong

For all $\varepsilon_1, \varepsilon_2 \in]0,1[$, $\varepsilon_1 < \varepsilon_2$, $n \geq 3$,

$$\begin{aligned}
|A_{\varepsilon_1} - A_{\varepsilon_2}| &= \int\limits_{B(x,\varepsilon_2)\setminus B(x,\varepsilon_1)} |\Phi(x-y)| \underbrace{|f(y)|}_{\geq M} \, \mathrm{d}y \\
&\leq M \int\limits_{B(x,\varepsilon_2)\setminus B(x,\varepsilon_1)} |\Phi(x-y)| \, \mathrm{d}y \\
&= M \int\limits_{B(x,\varepsilon_2)\setminus B(x,\varepsilon_1)} \underbrace{|\Phi(z)|}_{C_n|z|^{n-2}} \, \mathrm{d}z \\
&= MC_n \int_{\varepsilon_1}^{\varepsilon_2} \int_{\partial B(0,r)} r^{2-n} \, \mathrm{d}S\mathrm{d}r \\
&= MC_n \int_{\varepsilon_1}^{\varepsilon_2} r^{2-n} \underbrace{|\partial B(0,r)|}_{=n\alpha_n r^{n-1}} \, \mathrm{d}r \\
&= \bar{M} \int_{\varepsilon_1}^{\varepsilon_2} r \, \mathrm{d}r = \bar{M}\frac{1}{2}(\varepsilon_2^2 - \varepsilon_1^2) \\
&< \frac{1}{2}\varepsilon_2^2.
\end{aligned}$$

Thus we have a Cauchy sequence.

Theorem 4.6 (Solution of Poisson's equation)**.** *Let $f \in C_0^2(\mathbb{R}^n)$. Define u by*

$$u(x) = \lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^n\setminus B(x,\varepsilon)} \Phi(x-y)f(y)dy.$$

Then

(i) $u \in C^2(\mathbb{R}^n)$

(ii) $-\Delta u(x) = f(x)$ *for all* $x \in \mathbb{R}^n$.

"Proof".

$$\begin{aligned}
u(x) &= \int_{\mathbb{R}^n} \Phi(x-y)f(y) \, \mathrm{d}y \\
\partial u(x) &\overset{(!)}{=} \int_{\mathbb{R}^n} \partial_{x_i}\Phi(x-y)f(y) \, \mathrm{d}y \\
\Delta u(x) &= \sum_{i=1}^{n} \partial_{x_i x_i} u(x) = \int_{\mathbb{R}^n} \sum \underbrace{\partial_{x_i}\partial_{x_i}\Phi(x-y)}_{\text{"}=-\delta_x\text{" (!)}} f(y) \, \mathrm{d}y \\
&= -f(x).
\end{aligned}$$

□

The real proof. Proof of (i). u is continuous: Fix $x \in \mathbb{R}^n$. For all $\varepsilon > 0$, we wish to find $\delta = \delta(\varepsilon, x) > 0$ such that $|f(z) - f(x)| < \varepsilon$ if $|z - x| < \delta$.

$$\begin{aligned} |u(z) - u(x)| &\leq \int_{\mathbb{R}^n} |\Phi(y)|\,|f(z-y) - f(x-y)| \ \mathrm{d}y \quad (\mathbb{R}^n \text{ can be replaced}) \\ &= \int_{B(0,R+|x|+\delta)} |\Phi(y)|\,\underbrace{|f(z-y) - f(x-y)|}_{<\frac{\varepsilon}{M} \text{ for some } \delta \text{ since } f \text{ is uniformly continuous}} \ \mathrm{d}y \\ &< \frac{\varepsilon}{M} \underbrace{\int_{B(0,R+|x|+\delta)} |\Phi(y)| \ \mathrm{d}y}_{=:M} \\ &= \varepsilon \end{aligned}$$

when δ is chosen to be small in enough.

$u \in C^1(\mathbb{R}^n)$:

$$\frac{u(x+he_i) - u(x)}{h} = \int_{\mathbb{R}^n} \Phi(y) \underbrace{\frac{f(x+he_i) - f(x)}{h}}_{\text{has compact support for all } h} \ \mathrm{d}y$$

Since for all $\varepsilon > 0$, there exists $h_0 = h_0(\varepsilon) > 0$ such that $\dfrac{f(x+he_i-y) - f(x-y)}{h} - \partial_{x_i} f(x-y) < \varepsilon$ if $|h| < h_0$, we have

$$\partial_{x_i} u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i} f(x-y) \ \mathrm{d}y.$$

Similarly,

$$\partial_{x_i}\partial_{x_j} u(x) = \int_{\mathbb{R}^n} \Phi(y) \partial_{x_i}\partial_{x_j} f(x-y) \ \mathrm{d}y$$

so $u \in C^2(\mathbb{R}^n)$. $\square_{\text{(i)}}$

Proof of (ii).

$$\begin{aligned} \Delta u(x) &= \int_{\mathbb{R}^n} \Phi(y) \Delta_x f(x-y) \ \mathrm{d}y \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) \ \mathrm{d}y \\ &=: \lim_{\varepsilon \to 0} I_\varepsilon. \end{aligned}$$

Figure 4: Iso Integraali (The Great Integral) K [6]

$$
\begin{aligned}
I_\varepsilon &= \int_{\mathbb{R}^n\setminus B(0,\varepsilon)} \Phi(y) \underbrace{\Delta_y}_{!} f(x-y)\,\mathrm{d}y \quad (\mathbb{R}^n \text{ can be replaced with } B(0,R)) \\
&\overset{\text{i.p.b}}{=} \underbrace{\int_{\partial(0,R)} \Phi(y)\frac{\partial}{\partial\nu} f(x-y)\,\mathrm{d}S(y)}_{:=I_\varepsilon^1} + \underbrace{\int_{\partial B(0,\varepsilon)} -\Phi(y)\frac{\partial}{\partial\nu} f(x-y)\,\mathrm{d}S(y)}_{=:I_\varepsilon^2} \\
&\quad \underbrace{- \int\limits_{B(0,R)\setminus B(0,\varepsilon)} \nabla_y\Phi(y)\cdot\nabla_y f(x-y)\,\mathrm{d}y}_{=:I_\varepsilon^3}.
\end{aligned}
$$

$I_\varepsilon^1 = 0$ when R is chosen to be big enough. Also,

$$
\begin{aligned}
\left|I_\varepsilon^2\right| &\le \int_{\partial B(0,\varepsilon)} \underbrace{|\Phi(y)|}_{\substack{\text{constant on}\\ \partial B(0.\varepsilon)}} \left|\frac{\partial}{\partial\nu}\right| \underbrace{|f(x-y)|}_{\le M}\,\mathrm{d}S(y) \\
&= \underbrace{c_n\varepsilon^{n-1}}_{=|\partial B(0,\varepsilon)|} \begin{cases} M\frac{1}{n(n-2)\alpha_n}\varepsilon^{2-n}, & n\ge 3\\ M\frac{1}{2\pi}\log\frac{1}{\varepsilon}, & n=2 \end{cases} \\
&\le \begin{cases} c_n' M\varepsilon, & n\ge 3\\ M\varepsilon\log\frac{1}{\varepsilon}, & n=2 \end{cases} \\
&\overset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}
$$

$$
I_\varepsilon^3 \overset{\text{i.b.p}}{=} \underbrace{-\int_{\partial B(0,\varepsilon)} \frac{\partial}{\partial\nu}\Phi(y) f(x-y)\,\mathrm{d}S(y)}_{=:K} + \int_{B(0,R)\setminus B(0,\varepsilon)} \underbrace{\Delta\Phi(y)}_{=0} f(x-y)\,\mathrm{d}y.
$$

For $n\ge 3$ $\Phi(y) = \dfrac{1}{n(n-2)\alpha_n}|y|^{2-n}$, $\dfrac{\partial}{\partial\nu}\Phi(y) = \nabla\Phi(y)\cdot\nu(y)$ and $\nu(y) = -\dfrac{y}{|y|}$. By calculation

$$
\begin{aligned}
\nabla\Phi(y)\cdot\nu(y) = -(-1)\cdot\frac{|y|^{2-n}}{n\alpha_n|y|} &= \frac{|y|^{-n+1}}{n\alpha_n} \\
&= \frac{1}{n\alpha_n}\varepsilon^{-n+1}.
\end{aligned}
$$

So,

$$\begin{aligned} K &= -\frac{1}{n\alpha_n}\varepsilon^{-n+1}\int_{\partial B(0,\varepsilon)} f(x-y)\ \mathrm{d}S(y) \\ &= \fint_{\partial B(0,\varepsilon)} -f(x-y)\ \mathrm{d}S(y) \\ &\overset{\varepsilon\to 0}{\longrightarrow} -f(x). \end{aligned}$$

$\square_{\text{(ii)}}$

$\square$

5 Harmonic functions

Theorem 5.1 (Mean Value Theorem). *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^2(\Omega)$. The following are equivalent:*

(i) $\Delta u = 0$ *in* Ω,

(ii) $u(x) = \fint_{\partial B(x,r)} u(y)\ dS(y)$ *for all* $B(x,r) \subset\subset \Omega$ *and*

(iii) $u(x) = \fint_{B(x,r)} u(y)\ dy$ *for all* $B(x,r) \subset\subset \Omega$.

Proof. *(i) ⇒ (ii).* Fix a point $x \in \Omega$ and define

$$\varphi(r) = \fint_{\partial B(x,r)} u(y)\ \mathrm{d}S(y)$$

for all $0 < r < \operatorname{dist}(x, \partial\Omega)$. It then suffices to show that

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = 0,$$

since

$$\lim_{r\to 0}\fint_{\partial B(x,r)} u(y)\ \mathrm{d}S(y) = u(x).$$

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}r} &\overset{\text{Ex.}}{=} \fint_{\partial B(x,r)} \nabla u(\underbrace{x+ry}_{=:z})\cdot y\ \mathrm{d}S(y) \\ &\overset{\substack{\text{Chamge of}\\ \text{variables}}}{=} \int_{\partial B(x,r)} \frac{\nabla u(z)\cdot\frac{z-x}{r}}{|\partial B(0,1)|}\cdot\frac{1}{r^{n-1}}\ \mathrm{d}S(z) \\ &= \underbrace{\frac{1}{r^n\,|\partial B(0,1)|}}_{=|\partial B(x,r)|}\int_{\partial B(x,r)} \nabla u(z)\cdot\nu(z)\ \mathrm{d}S(z) \\ &\overset{\text{i.b.p}}{=} \int \underbrace{\Delta u(z)}_{=0}\ \mathrm{d}z = 0. \end{aligned}$$

$\square_{\text{(i)} \Rightarrow \text{(ii)}}$

(ii) ⇒ (i). By the previous calculations $\square_{\text{(ii)} \Rightarrow \text{(i)}}$

(ii) ⇒ (iii). Fix $\overline{B(x,r)} \subset \Omega$. Then

$$\begin{aligned}
⨍_{B(x,r)} u(y)\ \mathrm{d}y &= \frac{1}{|B(x,r)|}\int_{B(x,r)} u(y)\ \mathrm{d}y \\
&= \frac{1}{|B(x,r)|}\int_0^r \left[\int_{\partial B(x,t)} u(y)\ \mathrm{d}S(y)\right] \mathrm{d}t \\
&= \frac{1}{|B(x,r)|}\int_0^r |B(x,t)| \underbrace{\left[⨍_{\partial B(x,t)} u(y)\ \mathrm{d}S(y)\right]}_{=u(x)} \mathrm{d}t \\
&= \frac{u(x)}{|B(x,r)|}\int_0^r |\partial B(x,t)|\ \mathrm{d}t \\
&= u(x).
\end{aligned}$$

$\square_{\text{(ii)} \Rightarrow \text{(iii)}}$

(iii) ⇒ (ii). As above. $\square_{\text{(iii)} \Rightarrow \text{(ii)}}$

□

Remark 5.2 (Additional fun). Define

$$\varphi(r) = \frac{r\int_{B(x,r)} |\nabla u|^2\ \mathrm{d}y}{\int_{\partial B(x,r)} u^2\ \mathrm{d}S(y)}$$

for all $x \in \Omega$, $0 < r < \operatorname{dist}(x, \partial\Omega)$. (Compare φ with entropy.) Then $\varphi'(r) \geq 0$, if $\Delta u = 0$ in Ω. Also

$$\varphi(r) \to N(x)\ (\in \mathbb{Z}^+) \text{ as } r \to 0^+.$$

The integer $N(x)$ is called the *frequency* of u at the point x.

Exercise: Calculate $N(0)$ for $u = r^k \cos(k\theta)$.

Theorem 5.3. *If $u \in C(\Omega)$ and satisfies the property (ii) of Mean Value Theorem, then $u \in C^\infty\Omega)$. In particular, u is harmonic in Ω.*

Proof. To be proven later. □

5.1 Convolution and smoothing

Definition 5.4 (η-mollifier)**.** Define $\eta : \mathbb{R}^n \to \mathbb{R}$ by

$$\eta(x) = \begin{cases} C\exp\left(\dfrac{1}{|x|^2 - 1}\right) & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and set C such that $\int_{\mathbb{R}^n} \eta\ \mathrm{d}x = 1$. Call it *$\eta$-mollifier*[4] or simply *mollifier*.

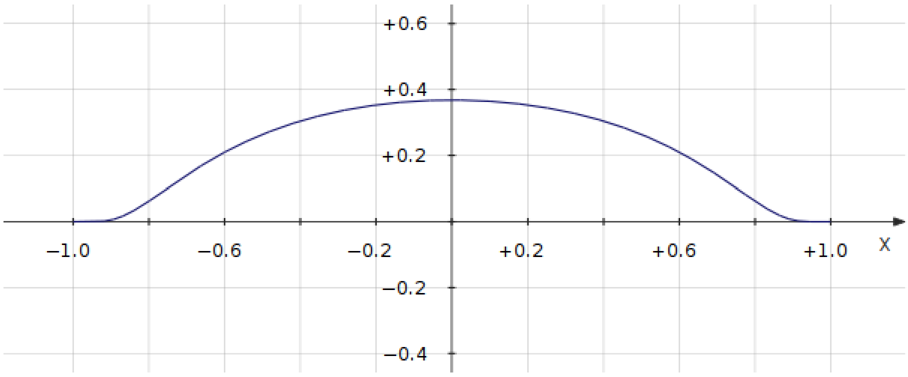

Figure 5: Plot of $\exp\left(\frac{1}{x^2-1}\right)$ by KmPlot

Exercise: Show that $\eta \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{spt}\eta = \overline{B(0,1)}$.

Definition 5.5 (η_ε)**.** For $\varepsilon > 0$ we set

$$\eta_\varepsilon(x) = \frac{1}{\varepsilon^n}\eta\left(\frac{x}{\varepsilon}\right).$$

Then $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{spt}\eta_\varepsilon = \overline{B(0,\varepsilon)}$.

Definition 5.6 (Mollification, $*$, f_ε)**.** Let $f \in C(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a domain and let $\varepsilon > 0$. Then the *mollification* or *smoothing* of f by η_ε is defined as

$$\begin{aligned} f_\varepsilon(x) := \eta_\varepsilon * f(x) &:= \int_\Omega \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y \\ &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)f(y)\,\mathrm{d}y, \end{aligned}$$

where $x \in \Omega_\varepsilon$ and

$$\Omega_\varepsilon := \{y \in \Omega : \operatorname{dist}(y,\partial\Omega) > \varepsilon\}.$$

Theorem 5.7 ("Mollifier Theorem")**.** *Let $f \in C(\Omega)$. Then*

(i) $f_\varepsilon \in C^\infty(\Omega_\varepsilon)$ for all $\varepsilon > 0$ and

(ii) $f_\varepsilon \to f$ uniformely on compact subsets of Ω.

Proof. Proof left as an exercise. □

Definition 5.8 (Smoothing by local averaging)**.** Define

$$\tilde{\eta}(x) := \frac{\chi_{B(0,1)}}{|B(0,1)|}$$

[4] η is just one of possible mollifiers. If you're new to mollifiers, read the Wikipedia page.

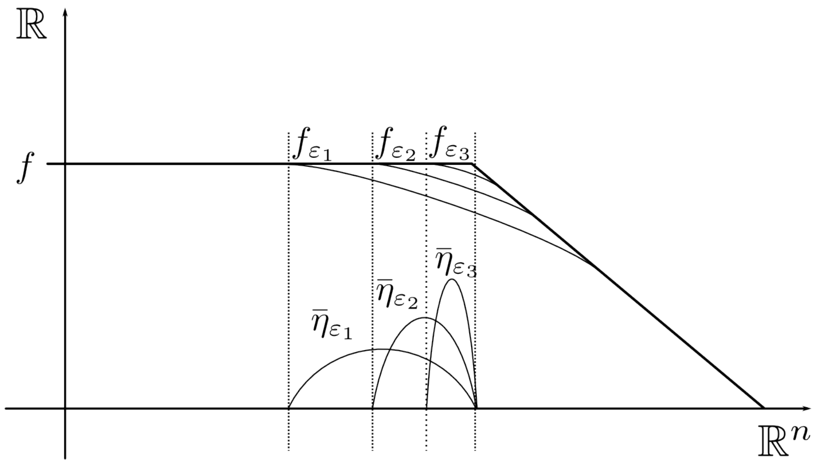

Figure 6: Smoothing of a function f by some $\overline{\eta}_{\varepsilon_i}$

and $\tilde{\eta}_\varepsilon(x) = \frac{1}{\varepsilon^n}\tilde{\eta}\left(\frac{x}{\varepsilon}\right)$ for all $\varepsilon > 0$. Then $\int_{\mathbb{R}^n} \tilde{\eta}_\varepsilon \, \mathrm{d}x = 1$ for all $\varepsilon > 0$.

Also, define

$$\tilde{f}_\varepsilon := \tilde{\eta}_\varepsilon * f := \fint_{B(x,\varepsilon)} f(y) \, \mathrm{d}y = \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f(y) \, \mathrm{d}y$$

and call it *smoothing by local averaging.*

5.2 Properties of harmonic fuctions

Theorem 5.9. *Suppose that $u \in C(\Omega)$ satisfies the Mean Value Theorem property*

$$u(x) = \fint_{\partial B(x,r)} u(y) \; dS(y)$$

for all $B(x,r) \subset \Omega$. Then $u \in C^\infty(\Omega)$ (and u is harmonic).

Corollary 5.10. *Harmonic functions are C^∞.*

Proof of the Theorem 5.9. Let

$$u_\varepsilon(x) := \eta_\varepsilon * u(x)$$

for all $x \in \Omega_\varepsilon$ and $\varepsilon > 0$. We first claim that $u(x)_\varepsilon = u(x)$ for all $x \in \Omega_\varepsilon$. By Mollifier Theorem part (ii), $u \in C^\infty(\Omega)$.

$$\begin{aligned} u_\varepsilon(x) &= \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)u(y) \, \mathrm{d}y \\ &= \int_0^\varepsilon \left[\int_{\partial B(x,t)} \eta_\varepsilon(x-y)u(y) \, \mathrm{d}S(y)\right] \, \mathrm{d}t \\ &= \int_0^\varepsilon \frac{1}{\varepsilon^n}\left[\int_{\partial B(x,t)} \eta\left(\frac{x-y}{\varepsilon}\right) u(y) \, \mathrm{d}S(y)\right] \\ &=: I \end{aligned}$$

Figure 7: Integral Cat [7]

Define $h(x)$ such that $\eta(x) = h(|x|)$.

Then

$$\begin{aligned} I &= \int_0^\varepsilon \frac{h\left(\frac{t}{\varepsilon}\right)}{\varepsilon^n} \underbrace{\int_{\partial B(x,t)} u(y)\ \mathrm{d}S(y)}_{=u(x)|\partial(x,t)|}\ \mathrm{d}t \\ &= u(x) \int_0^\varepsilon \frac{h\left(\frac{t}{\varepsilon}\right)}{\varepsilon^n} |\partial(x,t)|\ \mathrm{d}t \\ &= u(x) \int_{\mathbb{R}^n} \eta_\varepsilon\ \mathrm{d}x = u(x). \end{aligned}$$

□

Corollary 5.11. *u is harmonic $\Rightarrow \partial^\alpha u$ is harmonic.*

Theorem 5.12 (Weak and Strong Maximum Principle)**.** *Let Ω be a bounded domain. Let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a harmonic function. Then*

(i) $\max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$ and

(ii) if there exists $x_0 \in \Omega$ such that $u(x_0) = \max\limits_{\overline{\Omega}} u$ then $u(x) = u(x_0)$ for all $x \in \overline{\Omega}$.

Proof. It suffices to show (ii). Because $x_0 \in \Omega$, there exists $r > 0$ such that $\overline{B(x_0, r)} \subset \Omega$. Then

$$u(x_0) = ⨍_{B(x_0,r)} \underbrace{u(y)}_{\leq M}\ \mathrm{d}y = M := \max_{\overline{\Omega}} u.$$

So $u(y) = M$ inside $B(x_0, r)$. Therefore $u(x) = M$ for all $x \in \Omega$. □

Remark 5.13. Since $\Delta u = 0 \iff \Delta(-u) = 0$, similar minimum principle can be derived for the harmonic functions.

Theorem 5.14 (Uniqueness of a solution to Dirichlet problem)**.** *Let Ω be a domain, $g \in C(\partial\Omega)$ and $f \in C(\Omega)$. Then there exists at most one solution to the* Dirichlet problem

$$\begin{cases} \Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Proof. Let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ solve the Dirichlet problem. Set

$$w := u - v.$$

Then $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and

$$\begin{cases} \Delta w &= f \quad \text{in } \Omega \\ w &= 0 \quad \text{on } \partial\Omega. \end{cases}.$$

Thus $w \equiv 0$, by the Weak Maximum (and Minimum) Principle. □

Theorem 5.15 (Local estimates for harmonic functions)**.** *Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$. Then*

$$|D^\alpha u(x_0)| \leq \frac{c_k}{r^{n+k}} \int_{B(x_0,r)} |u(y)| \ dy$$

for all $\overline{B(x_0,r)} \subset \Omega$. Here $k = |\alpha|$ and $c_k = \dfrac{(2^{n+1}nk)^k}{|B^n(0,1)|}$.

Proof. $k = 0$.

$$\begin{aligned} |u(x_0)| &\overset{\text{M.V.T}}{=} \left| \frac{1}{\alpha_n r^n} \int_{B(x_0,r)} u(y) \ \mathrm{d}y \right| \\ &\leq \frac{1}{\alpha_n r^n} \int_{B(x_0,r)} |u(y)| \ \mathrm{d}y \end{aligned}$$

$\square_{k\,=\,0}$

$k = 1$. Use the Mean Value Theorem for $\partial_{x_i} u$:

$$\begin{aligned} |\partial_{x_i} u(x_0)| &\overset{\text{M.V.T}}{=} \left| \frac{1}{\alpha_n \left(\frac{r}{2}\right)^n} \int_{B(x_0,\frac{r}{2})} \partial_{x_i} u(y) \ \mathrm{d}y \right| \\ &\overset{\text{i.b.p}}{\leq} \frac{1}{\alpha_n \left(\frac{r}{2}\right)^n} \int_{\partial B(x_0,\frac{r}{2})} \left| u(y) \cdot \nu^i(y) \right| \ \mathrm{d}S(y) \ (\left|\nu^i\right| \leq 1) \\ &\leq \frac{1}{\alpha_n \left(\frac{r}{2}\right)^n} \int_{\partial B(x_0,\frac{r}{2})} |u(y)| \ \mathrm{d}S(y). \end{aligned}$$

Using the estimate from the case $k = 0$, obtain the claim. $\square_{k\,=\,1}$

$k \geq 2$. By induction, in a similar fashion.[5] $\square_{k\,\geq\,2}$

□

Theorem 5.16 (Analyticity of harmonic functions)**.** *Let u be a harmonic function in Ω. Then u is a* weak analytic *in Ω. (That is, it can be locally expressed as a convergent power series).*

[5]"Yeah! It seems to work. I'm too tired to prove anything so let's say we used induction." —Heikki

Proof. To show: For all x_0, u can be represented by a convergent power series in a neighbourhood of x_0. Taylor formula: For some $0 \leq t \leq 1$,

$$\begin{aligned} u(x) &= u(x_0) + \sum_{k=1}^{N-1} \sum_{|\alpha|=k} \frac{1}{\alpha!} D^\alpha u(x_0)(x-x_0)^\alpha \\ &\quad + \underbrace{\sum_{|\alpha|=N} \frac{1}{\alpha!} D^\alpha u(\overbrace{x_0 + t(x-x_0)}^{=:y})(x-x_0)^\alpha}_{=:R_N}, \end{aligned}$$

where $y^\alpha = y_1^{\alpha_1} \cdot y_2^{\alpha_2} \cdots y_n^{\alpha_n}$ and $\alpha! = \alpha_1! \cdot \alpha_2! \cdots \alpha_n!$. Note the following $k = 1$: $\nabla u(x_0) \cdot (x - x_0)$, $k = 2$: $\frac{1}{2}\left(D^2 u(x_0)(x-x_0)\right) \cdot (x-x_0)$. Then

$$u(x) = u(x_0) + \nabla u(x_0) \cdot (x-x_0) + \frac{1}{2}\left(D^2 u(x_0)(x-x_0)\right) \cdot (x-x_0) + \ldots$$

Let $r = \frac{1}{4}\operatorname{dist}(x_0, \partial\Omega) > 0$. Let $y = x_0 + t(x - x_0)$. Then $|y - x_0| \leq t\,|x - x_0| < r$. Now by Local Estimate Theory, for all $|\alpha| = N$,

$$\begin{aligned} |D^\alpha u(y)| &\leq \frac{(2^{n+1} n\,|\alpha|)^{|\alpha|}}{\alpha(n) r^{n+|\alpha|}} \int_{B(y,r)} |u(z)|\ \mathrm{d}z \leq \frac{c_N}{r^{n+N}} \underbrace{\int_{B(x_0,2r)} |u(z)|\ \mathrm{d}z}_{=:M} \\ &= \frac{c_N}{r^{n+N}} M. \end{aligned}$$

Here $\alpha(n) = |B^n(0,1)|$. Therefore

$$|R_N(x)| \leq \sum_{|\alpha|=N} \frac{|D^\alpha u(y)|}{\alpha!} \leq \sum_{|\alpha|=N} \frac{c_N M}{r^{n+N} 2^N} \overset{N\to\infty}{\longrightarrow} 0.$$

□

Theorem 5.17 (Liouville's Theorem)**.** *Suppose that u is a bounded harmonic function in $\mathbb{R}^n$. Then u is a constant.*

Proof. For all $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} |\nabla u(x_0)| &\leq \frac{c_n}{r^{n+1}} \int_{B(x_0,r)} \underbrace{|u(y)|}_{\leq M}\ \mathrm{d}y \\ &\leq \frac{M c_N \alpha_n r^n}{r^{n+1}} \overset{r\to\infty}{\longrightarrow} 0 \end{aligned}$$

for all $B(x_0, r) \subset \mathbb{R}^n$. Thus $|\nabla u(x_0)| = 0$ for all x_0. Therefore u is constant in $\mathbb{R}^n$. □

Theorem 5.18. *Let u be a harmonic function such that*

$$\int_{\mathbb{R}^n} |u(y)|\ dy < \infty.$$

Then $u \equiv 0$ in $\mathbb{R}^n$.

Proof. For all $x_0 \in \mathbb{R}^n$

$$|u(x_0)| \leq \frac{1}{\alpha_n r^n} \int_{B(x_0,r)} |u(y)| \, \mathrm{d}y \overset{r\to\infty}{\longrightarrow} 0.$$

□

Theorem 5.19. *Let u be a harmonic function in $\mathbb{R}^n$ such that it is bounded from below or above. Then u is a constant.*

Theorem 5.20 (Harnack's Inequality)**.** *Let $u \in C^2(\Omega)$ be a harmonic function and $u(x) \geq 0$ for all $x \in \Omega$. Then there exists a positive constant $C = C(n)$ such that*

$$\max_{B(x,r)} u \leq C \min_{B(x,r)} u$$

for all $B(x, 5r) \subset \Omega$. Here $C = 4^n$ will do.[6]

Proof. Fix $B(x, 5r) \subset \Omega$. For all $y, z \in \overline{B(x,r)}$

$$\begin{aligned} u(y) &= \fint_{B(y,4r)} u(w) \, \mathrm{d}w = \frac{1}{\alpha_n (4r)^n} \int_{B(y,4r)} u(w) \, \mathrm{d}w \\ &\geq \frac{1}{4^n} \frac{1}{\alpha_n r^n} \int_{B(z,r)} u(w) \, \mathrm{d}w \\ &= \frac{1}{4^n} \fint_{B(z,r)} u(w) \, \mathrm{d}w = \frac{1}{4^n} u(z), \end{aligned}$$

since $u \geq 0$ and $B(y, 4r) \supset B(z, r)$. Thus for all $y, z \in \overline{B(x,r)}$,

$$\frac{1}{4^n} u(z) \leq u(y) \leq 4^n u(z).$$

□

Corollary 5.21. *Let u be a non-negative harmonic function in Ω. Then for all $\Omega' \subset\subset \Omega$ there exists $C = C(\Omega', \Omega)$ such that*

$$\max_{\Omega'} u \leq C \min_{\Omega'} u.$$

Remark 5.22. For $B(x, 2r) \subset \Omega$, $\overline{B(x,r)} \subset \bigcup_{y \in B(x,r)} B\left(y, \frac{r}{5}\right)$. Since $\overline{B(x,r)}$ is compact, there exists $N \in \mathbb{N}$ and $y_1, \ldots, y_N$ such that $\overline{B(x,r)} \subset \bigcup_{i=1}^{N} B\left(y_i, \frac{r}{5}\right)$. By Harnack's inequality

$$\max_{B\left(y_i, \frac{r}{5}\right)} u \leq 4^n \min_{B\left(y_i, \frac{r}{5}\right)} u,$$

since $B(y_i, r) \subset B(x, 2r) \subset \Omega$.

[6]"Actually, it seems, you can pick $C = 2^n$ and repeat the proof, but again I'm too lazy to do that." —Heikki

Proof of the Corollary follows from here:

Proof of the Corollary 5.21. Let Ω' be such that $\overline{\Omega'} \subset \Omega$ is compact. Set

$$r := \frac{1}{100} \operatorname{dist}(\Omega', \partial\Omega) > 0.$$

Then $\overline{\Omega'} \subset \cup_{y \in \overline{\Omega'}} B(y, r)$. Since $\overline{\Omega'}$ compact, there exists $N \in \mathbb{N}$ and $y_1, \ldots, y_N$ such that

$$\overline{\Omega'} \subset \cup_{i=1}^{N} B(y_i, r).$$

For all $i = 1, \ldots, N$,

$$\max_{B(y_i,r)} u \leq 4^n \min_{B(y_i,r)} u,$$

since $B(y_i, 5r) \subset \Omega$. Therefore Corollary is proved with $C = (4^n)^N$. □

Now, using Harnack's inequality, prove the Theorem 5.19:

Proof of the Theorem 5.19. Let u be bounded below. Then there exists $M \in \mathbb{R}$ such that $M \leq u(x)$ for all $x \in \mathbb{R}^n$. Let

$$v(x) := u(x) - M.$$

Then v is harmonic and $v(x) \geq 0$ for all $x \in \mathbb{R}^n$. Now by Harnack's inequality,

$$\begin{aligned} \max_{B(0,r)} v &\leq 4^n \min_{B(0,r)} v \\ &\leq 4^n v(0) =: \overline{M} \end{aligned}$$

for all $B(0, 5r) \subset \mathbb{R}^n$. Thus $v(x) \leq \overline{M}$ for all $x \in \mathbb{R}^n$, so v is bounded. By Liouville's Theorem, $v(x)$ is a constant. Therefore, $u(x)$ is a constant also.

Proof for a function bounded above in same fashion. □

Theorem 5.23 (Very Strong Maximum Principle)**.** *Let $u \geq 0$ in $\Omega \subset \mathbb{R}^n$ be a harmonic function. Then there exists $C = C(n, \delta) > 0$ such that*

$$\max_{B(x,\delta r)} u \leq C \min_{B(x,\delta r)} u$$

for all $B(x, r) \subset \Omega$.

6 Green's function and Dirichlet problem

Remember the following theorem:

Theorem 6.1. *Let f be a function in $C^2(\mathbb{R}^n)$. Define*

$$u(x) = \int_{\mathbb{R}^n} \Phi(x - y) f(y) \; dy$$

where

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log|x|, & n=2 \\ \frac{1}{n(n-2)\alpha_n}|x|^{2-n}, & n\geq 3. \end{cases}$$

Then

$$-\Delta u = f \text{ in } \mathbb{R}^n. \tag{6.1}$$

Remark 6.2. $u+c$, $u+\overline{b}\cdot x$ are also solutions to (6.1), if u is.

Our goal is to solve the following Dirichlet problem:

$$\begin{cases} -\Delta u &= f & \text{in } \Omega \\ u &= g & \text{on } \partial\Omega. \end{cases}$$

Note that the uniqueness of the solution of a Dirichlet problem was already proven in Theorem 5.14.

Idea (bad): Define $V(x) = \int_\Omega \Phi(x-y)f(y)\,\mathrm{d}y$. Then $-\Delta V = f$ in Ω. But now might not be $V(x) = g$ on $\partial\Omega$!

Definition 6.3 (Φ^x)**.** Fix $x\in\Omega$. Assume that there exists a function $\Phi^x : \Omega \to \mathbb{R}$ that solves the problem

$$\begin{cases} \Delta_y \Phi^x(y) &= 0 & \text{for all } & y\in\Omega \\ \Phi^x(y) &= \Phi(x-y) & \text{for all } & y\in\partial\Omega. \end{cases} \tag{6.2}$$

See Example 6.11 for some cases of Φ^x. Next we will *assume* that we have solved (6.2) for Ω.

Definition 6.4 (Green's function)**.** *Green's function* in Ω is defined by

$$G^x(y) = G(x,y) = G_\Omega(x,y) = \Phi(x-y) - \Phi^x(y).$$

Remark 6.5.

- $G(x,y) = 0$ for all $y\in\partial\Omega$.
- $\Delta_y G(x,y) = 0$ for all $y\neq x$.
- $G(y,x)\to\infty$ as $y\to x$.

Theorem 6.6 (Representation Formula using Green's function)**.** *Let* $u\in C^2(\Omega)\cap C^1(\overline{\Omega})$. *Then*

$$u(x) = -\int_{\partial\Omega} u(y)\frac{\partial G(x,y)}{\partial\nu}\,dS(y) - \int_\Omega G(x,y)\Delta_y u(y)\,dy.$$

Proof. Proof by calculation.

$$\begin{aligned}
I_\varepsilon &:= \int_{\Omega\setminus B(x,\varepsilon)} G(x,y)\Delta_y u(y)\ \mathrm{d}y \\
&= \int_{\partial(\Omega\setminus B(x,\varepsilon))} G(x-y)\frac{\partial u}{\partial\nu}(y)\ \mathrm{d}S(y) - \int_{\Omega\setminus B(x,\varepsilon)} \nabla G(x,y)\nabla u\ \mathrm{d}y \\
&= \int_{\partial(\Omega\setminus B(x,\varepsilon))} G(x-y)\frac{\partial u}{\partial\nu}(y)\ \mathrm{d}S(y) - \int_{\partial(\Omega\setminus B(x,\varepsilon))} \frac{\partial G(x,y)}{\partial\nu} u(y)\ \mathrm{d}S(y) \\
&\quad + \int_{\Omega\setminus B(x,\varepsilon)} \overbrace{\Delta G(x,y)}^{=0} u(y)\ \mathrm{d}y \\
&=: I_\varepsilon^1 - I_\varepsilon^2. \\
\left|I_\varepsilon^1\right| &\le \underbrace{\left|\int_{\partial\Omega} \overbrace{G(x,y)}^{=0}\frac{\partial u}{\partial\nu}(y)\ \mathrm{d}S(y)\right|}_{=0} + \underbrace{\int_{\partial B(x,\varepsilon)} \overbrace{|G(x,y)|}^{\le M+c\varepsilon^{2-n}} \overbrace{\left|\frac{\partial u}{\partial\nu}(y)\right|}^{\le K}\ \mathrm{d}S(y)}_{\le Kc\varepsilon^{n-1}(M+c\varepsilon^{2-n})\stackrel{\varepsilon\to 0}{\longrightarrow}0} \\
&\stackrel{\varepsilon\to 0}{\longrightarrow} 0. \\
I_\varepsilon^2 &= \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial\nu} u(y)\ \mathrm{d}S(y) + \int_{\partial B(x,\varepsilon)} \frac{\partial G(x,y)}{\partial\nu} u(y)\ \mathrm{d}S(y) \\
&=: I^{2,1} + I_\varepsilon^{2,2}. \\
I_\varepsilon^{2,2} &= \underbrace{\int_{\partial B(x,\varepsilon)} \frac{\partial\Phi(x-y)}{\partial\nu} u(y)\ \mathrm{d}S(y)}_{\stackrel{\varepsilon\to 0}{\longrightarrow}u(x)} - \underbrace{\int_{\partial B(x,\varepsilon)} \frac{\partial\Phi^x(y)}{\partial\nu} u(y)\ \mathrm{d}S(y)}_{\stackrel{\varepsilon\to 0}{\longrightarrow}0} \\
&\stackrel{\varepsilon\to 0}{\longrightarrow} u(x).
\end{aligned}$$

Therefore $I_\varepsilon = I_\varepsilon^1 - I_\varepsilon^2 \to -I^{2,1} - u(x)$ as $\varepsilon \to 0$. As a limit we have

$$\int_\Omega G(x,y)\Delta_y u(y)\ \mathrm{d}y = -\int_{\partial\Omega} \frac{\partial G(x,y)}{\partial\nu} u(y)\ \mathrm{d}S(y) - u(x).$$

□

Remark 6.7.

$$-\Delta u(y) = f(y) \in C(\Omega) \Longrightarrow \int_\Omega f(y)\ \mathrm{d}y = -\in_\Omega \Delta u(y)\ \mathrm{d}y = -\int \frac{\partial u}{\partial\nu}\ \mathrm{d}S(y) < \infty.$$

Theorem 6.8. *Suppose $u \in C(\Omega) \cap C^1(\overline{\Omega})$ is a solution to*

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega. \end{cases}$$

Then

$$u(x) = -\int_{\partial\Omega} \frac{\partial G(x,y)}{\partial\nu}\ dS(y) + \int_\Omega G(x,y)f(y)\ dy.$$

Theorem 6.9. *For all $x, y \in \Omega$ such that $x \neq y$,*

$$G(x,y) = G(y,x).$$

Proof. Fix $x, y \in \Omega$, $x \neq y$. Write

$$v(z) = G(x,z) \text{ and } w(z) = G(y,z)$$

for all $z \in \Omega$.

Idea:

$$\begin{aligned} -\Delta_z v(z) = \delta_x \quad &\Rightarrow \quad w(x) = \int_\Omega w(z)\delta_x = -\int_\Omega \Delta_z v(z) w(z) \\ -\Delta_z w(z) = \delta_y \quad &\Rightarrow \quad v(y) = \int_\Omega v(z)\delta_y = -\int_\Omega \Delta_z w(z) v(z) \end{aligned}$$

$\Rightarrow v(y) = w(x)$.

Recall, that $v(z) = w(z) = 0$ for all $z \in \partial\Omega$. Choose $\varepsilon > 0$ small enough, that $B(x,\varepsilon) \cap B(y,\varepsilon) = \emptyset$, $B(x,\varepsilon) \subset \Omega$, $B(y,\varepsilon) \subset \Omega$. Denote $\Omega' = \Omega \setminus (B(x,\varepsilon) \cup B(y,\varepsilon))$. Then

$$\begin{aligned} 0 &= \int_{\Omega'} v(z) \underbrace{\Delta_z w(z)}_{=0} - w(z) \underbrace{\Delta_z v(z)}_{=0} \, \mathrm{d}z \\ &= \int_{\partial\Omega'} v\frac{\partial w}{\partial \nu} - w\frac{\partial v}{\partial \nu} \, \mathrm{d}S(x) \\ &= \underbrace{\int_\Omega \ldots \, \mathrm{d}S(x)}_{=0} + \underbrace{\int_{\partial B(x,\varepsilon)} \ldots \, \mathrm{d}S(x)}_{=:I_\varepsilon^x} + \underbrace{\int_{\partial B(y,\varepsilon)} \ldots \, \mathrm{d}S(x)}_{=:I_\varepsilon^y} \\ &\overset{\varepsilon \to 0}{\longrightarrow} -w(x) + v(y), \end{aligned}$$

where

$$\begin{aligned}
I_\varepsilon^x &= \int_{\partial B(x,\varepsilon)} v\frac{\partial w}{\partial \nu}\,\mathrm{d}S(x) - \int_{\partial B(x,\varepsilon)} w\frac{\partial v}{\partial \nu}\,\mathrm{d}S(x) \\
&=: I_{\varepsilon,1}^x + I_{\varepsilon,2}^x. \\
\left|I_{\varepsilon,1}^x\right| &\le \int_{\partial B(x,\varepsilon)} \underbrace{|v(z)|}_{\le c\varepsilon^{2-n}+M} \underbrace{\left|\frac{\partial w}{\partial \nu}(z)\right|}_{\le M|x-y|}\,\mathrm{d}S(x)(z) \\
&\overset{\varepsilon\to 0}{\longrightarrow} 0. \\
|v(z)| &= |G(x,z)| \le |\Phi(z-x)| + |\Phi^x(z)| \\
&\le c\,|z-x|^{2-n} + M \\
&= c\varepsilon^{2-n} + M \text{ on } \partial B(x,\varepsilon). \\
&\overset{\varepsilon\to 0}{\longrightarrow} 0 \\
I_{\varepsilon,2}^x &= -\int_{\partial B(x,\varepsilon)} w(z)\frac{\partial \Phi(z-x)}{\partial \nu}\,\mathrm{d}s(z) + \underbrace{\int_{\partial B(x,\varepsilon)} w(z)\frac{\partial \Phi^x(z)}{\partial \nu}\,\mathrm{d}s(z)}_{\overset{\varepsilon\to 0}{\longrightarrow} 0} \\
&= -⨍_{\partial B(x,\varepsilon)} w(z)\,\mathrm{d}s(z) \\
&\overset{\varepsilon\to 0}{\longrightarrow} -w(x).
\end{aligned}$$

Calculate in same fashion and obtain $I_\varepsilon^y \overset{\varepsilon\to 0}{\longrightarrow} v(y)$.

□

We need bigger weapons in war against the equations, so:

Theorem 6.10 (3G Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$ be a bounded smooth domain. Then there exists $c = c(\Omega) > 0$ such that*

$$\frac{G(x,y)G(y,z)}{G(x,z)} \le c\left(|x-y|^{2-n} + |y-z|^{2-n}\right).$$

Proof. Proof left as an exercise. $\square_{\text{3G}}$

Example 6.11 (Φ^x)**. Case 1:** Let $\Omega = \mathbb{R}_+^n =: \{z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n : z_n > 0\}$. Then $\Phi^x(y) = \Phi(y - \overline{x})$, where $\overline{x} = (x_1, x_2, \dots, -x_n)$.
Case 2: Let $\Omega = B(0,1)$. Then $\Phi^x(y) = \Phi(|x|\,(y - \overline{x}))$, where $\overline{x} = \dfrac{x}{|x|^2}$. To be proven later.

Remark 6.12.

$$\begin{aligned}
G_{R^n_+} &= G(x,y) = \Phi(y-x) - \Phi(y-\overline{x}) \\
\frac{\partial G(x,y)}{\partial \nu} &= \nabla_y G(x,y) \cdot \underbrace{\nu(y)}_{(0,\dots,0,-1)} = -\frac{\partial}{\partial y_n} G(x,y) \\
&= \frac{1}{n\alpha_n} \left(\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-x|^n} \right) \\
&= -\frac{2x_n}{n\alpha_n |y-x|^n}.
\end{aligned}$$

Definition 6.13 (Poisson kernel for $\mathbb{R}^n_+$)**.** We call the function

$$K(x,y) = \frac{2x_n}{n\alpha(n)|y-x|^n}, \; x \in \mathbb{R}^n_+, \; y \in \partial\mathbb{R}^n_+$$

the *Poisson kernel for* $\mathbb{R}^n_+$.

Theorem 6.14. *Assume $g \in C(\partial\mathbb{R}^n_+)$ is bounded such that $|g(y)| \le M$ for all $y \in \partial\mathbb{R}^n_+$. Define*

$$u(x) = \int_{\partial\mathbb{R}^n_+} K(x,y) g(y) \; dS(y)$$

for all $x \in \mathbb{R}^n_+$ and $y = (y_1, \dots, y_{n-1}, 0)$. Then

(i) $u \in C^\infty(\mathbb{R}^n)$ *and* u *is bounded in* $\mathbb{R}^n_+$

(ii) $\Delta u = 0$ *in* $\mathbb{R}^n_+$

(iii) $\lim_{\substack{x \to y \\ x \in \mathbb{R}^n_+}} u(x) = g(y)$ *for all* $y \in \partial\mathbb{R}^n_+$.

Proof. Proof of (i). $\int_{\partial\mathbb{R}^n_+} K(x,y) \,\mathrm{d}S(y) = 1$ for all $x \in \mathbb{R}^n_+$. Unformal:

$$\begin{aligned}
\int_{\mathbb{R}^n_+} K(x,y) &= \int_{\mathbb{R}^n_+} \frac{\partial G(x,y)}{\partial \nu} \,\mathrm{d}S(y) \quad \nu = (0,\dots,0,-1) \text{ on } \partial\mathbb{R}^n_+ \\
&= -\int_{\mathbb{R}^n_+} \Delta G(x,y) \,\mathrm{d}y = 1,
\end{aligned}$$

since $-\Delta_y G(x,y) = \delta_x$.

Formal:

$$-\int_{\mathbb{R}^n_+ \setminus B(x,\varepsilon)} \Delta G(x,y) \,\mathrm{d}y = \dots \varepsilon \to 0 \dots = 1.$$

$$|u(x)| \le \int_{\partial R^n_+} K(x,y) \underbrace{|g(y)|}_{\le M} \,\mathrm{d}S(y) \le M \quad \forall\, x \in \mathbb{R}^n_+$$

$u \in C^\infty(\mathbb{R}^n_+)$:

$$D_x^\alpha u(x) = \int_{\partial\mathbb{R}^n_+} D_x^\alpha K(x,y) g(y) \, \mathrm{d}S(y).$$

$\square_{\text{(i)}}$

Proof of (ii).

$$\Delta_x u(x) = \int_{\partial\mathbb{R}^n_+} \underbrace{\Delta K(x,y)}_{=0} g(y) \, \mathrm{d}S(y)$$

$K(x,y) = -\partial_{x_n}\Phi(x-y)$ so K is harmonic with respect to x.

$$K(x,0) = \frac{2x_n}{n\alpha_n |x|^n}.$$

$\square_{\text{(ii)}}$

Proof of (iii). Fix $x_0 \in \partial\mathbb{R}^n_+$, $\varepsilon > 0$. Choose $\delta > 0$ so small that

$$|g(y) - g(x_0)| < \varepsilon,$$

if $|y - x_0| < \delta$, $y \in \partial\mathbb{R}^n_+$. Then if $|x - x_0| < \frac{\delta}{2}$, $x \in \mathbb{R}^n_+$,

$$\begin{aligned} |u(x) - g(x_0)| &\leq \int_{\mathbb{R}^n_+} K(x,y) \, |g(y) - g(x_0)| \, \mathrm{d}S(y) \\ &= \int_{|y-x_0|<\delta} K(x,y) \underbrace{|g(y) - g(x_0)|}_{<\varepsilon} \, \mathrm{d}S(y) \\ &\quad + \int_{|y-x_0|\geq\delta} K(x,y) \underbrace{|g(y) - g(x_0)|}_{\leq 2M} \, \mathrm{d}S(y) \\ &< \varepsilon + 2M \int_{|y-x_0|\geq\delta} K(x,y) \, \mathrm{d}S(y). \end{aligned}$$

$|y - x_0| \geq \delta$ implies $|y - x_0| \leq |y - x| + \underbrace{|x - x_0|}_{\frac{|y-x_0|}{2}}$ so $|y - x| \geq \frac{|y-x_0|}{2}$. Thus

$$K(x,y) = \frac{2x_n}{n\alpha_n |x-y|^n} \leq \frac{cx_n}{|y-x_0|^n}$$

and therefore

$$\begin{aligned}
\int\limits_{|y-x_0|\geq\delta} K(x,y)\,\mathrm{d}S(y) &\leq c \int\limits_{|y-x_0|\geq\delta} \frac{x_n}{|y-x_0|^n}\,\mathrm{d}S(y) \\
&= c\int_{|y|\geq\delta} \frac{x_n}{|y|^n}\,\mathrm{d}S(y) \\
&= cx_n \int_{|\overline{y}|>\delta} \frac{1}{|\overline{y}|^n}\,\mathrm{d}\overline{y} \\
&= cx_n \underbrace{\int\limits_{\delta}^{\infty} \underbrace{\int\limits_{\partial B(0,r)} \frac{1}{r^n}\,\mathrm{d}S(\overline{y})}_{=c(n)r^{n-2-n}\mathrm{d}r}}_{=c\delta^{-1}} \\
&= c\frac{x_n}{\delta} < \varepsilon,
\end{aligned}$$

if $|x_n| \leq |x-x_0| < \dfrac{\delta\varepsilon}{M}$. Here $\overline{y} = (y_1,\dots,y_{n-1})$. So if $|x-x_0| < \min\left(\frac{\delta}{2}, \frac{\delta\varepsilon}{2}\right)$, then

$$|u(x) - g(x_0)| < 2\varepsilon.$$

$\square_{\text{(iii)}}$

$\square$

Next we'll deduce the Φ^x for the unit ball. If $y \in \partial B(0,1)$, then

$$\begin{aligned}
\frac{|\overline{x}-y|^2}{|x-y|^2} &= \frac{\left|\frac{x}{|x|^2}-y\right|^2}{|x-y|^2} = \frac{\left\langle \frac{x}{|x|^2}-y, \frac{x}{|x|^2}-y\right\rangle}{\langle x-y, x-y\rangle} \\
&= \frac{\left\langle \frac{x}{|x|^2}, \frac{x}{|x|^2}\right\rangle - 2\left\langle \frac{x}{|x|}, y\right\rangle + \langle y,y\rangle}{\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle} \\
&= \frac{\frac{1}{|x|^2} - \frac{2}{|x|^2}\langle x,y\rangle + 1}{|x|^2 - 2\langle x,y\rangle + 1} = \frac{1}{|x|^2}.
\end{aligned}$$

Therefore $\dfrac{|\overline{x}-y|}{|x-y|} = \dfrac{1}{|x|}$.

Theorem 6.15 (Φ^x for the unit ball)**.** *Let $B = B(0,1) = \{z \in \mathbb{R}^n : |1| < 1\}$. Then*

$$\Phi^x(y) = \Phi\left((y-\overline{x})\cdot|x|\right)$$

is the solution to

$$\begin{cases} \Delta_y \Phi^x(y) = 0 & \text{for all } y \in B \\ \Phi^x(y) = \Phi(y-x) & \text{for all } y \in \partial B. \end{cases} \tag{6.3}$$

Proof.

$$\Phi^x(y) = |x|^{2-n} \frac{1}{n(n-2)\alpha_n} |y - \overline{x}|^{2-n} = |x|^{2-n} \Phi(y - \overline{x})$$

is harmonic in B.

$$\begin{aligned} \Phi^x(y) = \Phi\left((y - \overline{x}) \cdot |x|\right) &= \frac{1}{n(n-2)\alpha_n} |y - x|^{2-n} \\ &= \Phi(x - y), \end{aligned}$$

when $|x| = 1$, that is, on ∂B. □

Remark 6.16.

$$\begin{aligned} G(x, y) &= \Phi(y - x) - \Phi\left((y - \overline{x})\,|x|\right) \\ \frac{\partial G(x, y)}{\partial \nu} &= \Delta_y G(x, y) \cdot \nu(y) \\ &= \sum_{i=1}^{n} \frac{\partial G}{\partial y_i} \cdot y_i. \end{aligned}$$

$$\begin{aligned} \frac{\partial G}{\partial y_i}(x, y) &= \frac{\partial \Phi(y - x)}{\partial y_i} - \frac{\partial \Phi\left((y - \overline{x})\,|x|\right)}{\partial y_i} \\ &= -\frac{1}{n\alpha_n} \frac{y_i - x_i}{|y - x|^n} \\ &= -\frac{1}{n\alpha_n} \frac{y_i - \overline{x}_i}{|y - \overline{x}|^n} \\ &= -\frac{1}{n\alpha_n} \frac{y_i\,|x|^2 - \overline{x}_i}{|x - y|^n}. \end{aligned}$$

Therefore (exercise)

$$\frac{\partial G(x, y)}{\partial \nu} = -\frac{1}{n\alpha_n} \frac{1 - |x|^2}{|x - y|^n}.$$

This gives us the motivation for the next definition.

Definition 6.17 (Poisson kernel for the unit ball)**.** Let $B = B(0, 1) = \{z \in \mathbb{R}^n : |z| < 1\}$. For all $x, y \in B$, we define the *Poisson Kernel for B* by setting

$$K(x, y) = \frac{1}{n\alpha_n} \cdot \frac{1 - |x|^2}{|x - y|^n}.$$

Theorem 6.18. *Let $g \in C(\partial B(0, 1))$ be a fixed function. Define*

$$u(x) = \int_{\partial B(0,1)} K(x, y) g(y)\ dS(y)$$

for all $x, y \in B(0, 1)$. Then

(i) $u \in C^\infty(B(0,1)) \cap C(\overline{B}(0,1))$

(ii) $\Delta u = 0$ *in* $B(0,1)$

(iii) $u(x) = g(x)$ *for all* $y \in \partial B(0,1)$.

The same holds for $B(0,r)$ *with*

$$K(x,y) = \frac{r^2 - |x|^2}{n\alpha_n r\, |x-y|^n},$$

the Poisson kernel for $B(0,r) \subset \mathbb{R}^n$.

Theorem 6.19 (Energy method)**.** *There exists at most one solution* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *to*

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \quad u &= g \quad \text{on } \partial\Omega. \end{cases} \quad \textit{(Euler-Lagrange equation)} \tag{6.4}$$

Proof. Let $u_1, u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be two solutions to (6.4). Let $w = u_1 - u_2$. Then

$$\begin{cases} -\Delta w &= 0 \quad \text{in } \Omega \\ \quad w &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{6.5}$$

$$0 = \int_\Omega -\Delta w \cdot w \, \mathrm{d}x = \int_\Omega |\nabla w|^2$$

implying $|\nabla w|^2 = 0$ in Ω. Thus $w \equiv 0$ in Ω. □

Definition 6.20 (Energy)**.** We define *energy* by setting

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - uf \, \mathrm{d}x,$$

where u belongs to the admissible classical

$$K = \left\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : u = g \text{ on } \partial\Omega \right\}$$

where $f \in C(\overline{\Omega})$ and $g \in C(\partial\Omega)$ are given.

Theorem 6.21. *Assume that* $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *solves (6.4). Then*

(i)

$$I(u) = \min_{w \in K} I(w).$$

I.e. $I(u) \le I(w)$ *for all* $w \in K$.

(ii) Conversely, if $u \in K$ *is such that* $I(u) = \min_{w\in K} I(w)$, *then* u *is the solution of (6.4).*

Proof. Proof of (i). Fix $w \in K$. Let $v = w - u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. $v = 0$ on $\partial\Omega$. Now

$$\begin{aligned}\int_\Omega fv \, \mathrm{d}x &= \int_\Omega -\Delta u \cdot v \, \mathrm{d}x \\ &= \int_\Omega \nabla u \nabla v \, \mathrm{d}x \\ &= \int_\Omega \nabla u \left(\nabla w - \nabla u\right) \, \mathrm{d}x.\end{aligned}$$

We want to prove that $I(u) \leq I(w)$, that is

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u|^2 - uf \, \mathrm{d}x &\leq \frac{1}{2} |\nabla w|^2 - wf \, \mathrm{d}x \\ \Leftrightarrow \frac{1}{2}\int_\Omega |\nabla u|^2 - |\nabla w|^2 \, \mathrm{d}x &\leq -\int_\Omega vf \, \mathrm{d}x = \int_\Omega |\nabla u|^2 - \nabla u \nabla w \, \mathrm{d}x \\ &\Leftrightarrow 0 \leq \frac{1}{2}\int_\Omega |\nabla u - \nabla w|^2 \, \mathrm{d}x.\end{aligned}$$

$\square_{\text{(i)}}$

Proof of (ii). Fix $\varphi \in C_0^\infty(\Omega)$. For $t \in \mathbb{R}$ we define

$$w = u + t\varphi \in K.$$

We define

$$h(t) = I(u + t\varphi).$$

Then $h(0) \leq h(t) \Leftrightarrow I(u) \leq I(w)$ for all $t \in \mathbb{R}$ and

$$h'(t)|_{t=0} = 0.$$

$$\begin{aligned}0 = h(t) &= \frac{1}{2}\int_\Omega |\nabla(u + t\varphi)|^2 - (u + t\varphi)f \, \mathrm{d}x \\ h'(t)|_{t=0} &= \int_\Omega \nabla(u + t\varphi) \cdot \nabla\varphi - \varphi f \, \mathrm{d}x|_{t=0} \\ &= \underbrace{\int_\Omega \nabla u \cdot \nabla\varphi \, \mathrm{d}x}_{= -\int_\Omega \Delta u \varphi \, \mathrm{d}x} - \int_\Omega \varphi f \, \mathrm{d}x \\ &= \int_\Omega (-\Delta u - f)\varphi \, \mathrm{d}x\end{aligned}$$

implying $-\Delta u = f$ in Ω. $\square_{\text{(ii)}}$

$\square$

Theorem 6.22. *Let $u \in C^2(B(0,1)) \cap C^1(\overline{B(0,1)})$, $B(0,1) \subset \mathbb{R}^n$, $n \geq 3$ be a solution to*

$$\begin{cases} -\Delta u & = & |u|^{\frac{4}{n-2}} u & \text{in } B(0,1) \\ u & = & 0 & \text{on } \partial B(0,1). \end{cases}.$$

Then $u \equiv 0$ in $B(0,1)$.

Proof. Proof left as an exercise, because I can't make sens of the "proof" in the notes. ☐

7 Helmholtz's equation

Let $\Omega \subset \mathbb{R}^n$ be bounded C^1-domain. We aim to solve following problem (*Helmholtz equation*): Find $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\begin{cases} -\Delta u & = & \lambda u & \text{in } \Omega, \quad \lambda \in \mathbb{R} \\ u & = & 0 & \text{on } \partial\Omega. \end{cases} \tag{7.1}$$

Remark 7.1. If $\lambda \leq 0$, then $u \equiv 0$.

Proof.

$$\begin{aligned} \int_\Omega -\Delta u \cdot u \, \mathrm{d}x &= \int_\Omega \lambda u \cdot u \, \mathrm{d}x \\ \Leftrightarrow \underbrace{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}_{\geq 0} &= \underbrace{\lambda}_{\leq 0} \underbrace{\int_\Omega u^2 \, \mathrm{d}x}_{\geq 0}, \end{aligned}$$

which implies $u \equiv 0$ in Ω. ☐

Definition 7.2 (Eigenvalue, eigenfunction)**.** $\lambda \in \mathbb{R}$ is an *eigenvalue* of $-\Delta$ in Ω, if (7.1) has a non-trivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$. We call the solution u the *eigenfunction* corresponding to this eigenvalue λ.

Definition 7.3 (Rayleigh's quotient)**.** Let $w \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $w \not\equiv 0$ in Ω, $w = 0$ on $\partial\Omega$. We define

$$Q(w) = \frac{\int_\Omega |\nabla w|^2 \, \mathrm{d}x}{\int_\Omega w^2} \geq 0$$

and

$$m = \inf_w Q(w).$$

Theorem 7.4.

$$m \geq \frac{n^2}{4 \operatorname{diam}(\Omega)^2} > 0.$$

Theorem 7.5 (Poincaré inequation)**.**

$$\int_\Omega w^2 \ dx \leq \frac{4\,\mathrm{diam}(\Omega)^2}{n^2} \int_\Omega |\nabla w|^2 \ dx.$$

for all $w \in C^1(\overline{\Omega})$ such that $w = 0$ on $\partial\Omega$.

Remark 7.6. It suffices to prove that

$$\inf_{\substack{w \in C^1(\overline{\Omega}) \\ w=0 \text{ on } \partial\Omega \\ w \not\equiv 0}} Q(w) \geq \frac{n^2}{4\,\mathrm{diam}(\Omega)^2}.$$

Proof.

$$\begin{aligned} n \int_\Omega w^2 \ \mathrm{d}x &\overset{\text{i.p.b}}{=} \left| -\int_\Omega \nabla(w^2) \cdot x \right| \\ &= \left| -\int_\Omega \partial_{x_i} \left(w^2(x) \right) \cdot x_i \ \mathrm{d}x \right| \\ &\leq |-2| \int_\Omega |w| \underbrace{|\nabla w \cdot x|}_{\leq |\nabla w||x|} \ \mathrm{d}x. \end{aligned}$$

This implies

$$\begin{aligned} \int_\Omega w^2 \ \mathrm{d}x &\leq \frac{2\,\mathrm{diam}(\Omega)}{n} \int_\Omega |w|\,|\nabla w| \ \mathrm{d}x \\ &\leq \frac{1}{2} \int_\Omega w^2 \ \mathrm{d}x + \frac{2\,\mathrm{diam}(\Omega)^2}{n^2} \int_\Omega |\nabla w|^2 \ \mathrm{d}x. \end{aligned}$$

(Note that: $2ab \leq a^2 + b^2$ and put $a = \frac{w}{\sqrt{2}}$, $b = \sqrt{2}\frac{\mathrm{diam}(\Omega)}{n} |\nabla w|$.) Thus,

$$\int_\Omega w^2 \ \mathrm{d}x \leq \frac{4\,\mathrm{diam}(\Omega)^2}{n^2} \int_\Omega |\nabla w|^2 \ \mathrm{d}x.$$

□

Lemma 7.7. *If λ is an eigenvalue of $-\Delta$ in Ω, then $\lambda \geq m$.*

Proof. There exists $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $v \neq 0$ and $v = 0$ on $\partial\Omega$ and

$$-\Delta v = \lambda v.$$

Then

$$\begin{aligned} \int_\Omega |\Delta v|^2 = \int_\Omega -\Delta v \cdot v \ \mathrm{d}x &= \int_\Omega \lambda v \cdot v \ \mathrm{d}x \\ &= \lambda \int_\Omega v^2 \ \mathrm{d}x \end{aligned}$$

implying

$$\lambda = \frac{\int_\Omega |\nabla v|^2 \ \mathrm{d}x}{\int_\Omega v^2 \ \mathrm{d}x} \geq m.$$

□

Theorem 7.8. *Suppose that there is $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, $u = 0$ on $\partial\Omega$, $u \not\equiv 0$ in Ω, such that $Q(u) = m$. Then m is the smallest eigenvalue of $-\Delta$ (the* first eigenvalue, principle eigenvalue, *λ_1) and u is the* principle eigenfunction.

Proof. For all $\varphi \in C_0^\infty(\Omega)$, $t \in \mathbb{R}$, $w_t = u + t\varphi$. Let

$$h(t) = Q(w_t) = \frac{\int_\Omega |\nabla(u + t\varphi)|^2 \, \mathrm{d}x}{\int_\Omega |u + t\varphi|^2 \, \mathrm{d}x}.$$

Then $h'(t)|_{t=0} = 0$. Calculate and obtain

$$\underbrace{\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x}_{=\int_\Omega (-\Delta u)\varphi \, \mathrm{d}x} = \underbrace{\frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x}}_{=m} \int_\Omega u\varphi \, \mathrm{d}x$$

Then $\int_\Omega (-\Delta u - mu)\varphi \, \mathrm{d}x = 0$. Thus

$$-\Delta u - mu = 0$$

in Ω. □

Theorem 7.9. *There exists $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$, such that $u = 0$ on $\partial\Omega$, $u \not\equiv 0$ in Ω and*

$$Q(u) = \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega u^2 \, dx} = m.$$

Remark 7.10.

- $m = \lambda_1 = \lambda_1(\Omega) > 0$
- Let u be an eigenfunction to λ_1, then $Q(u) = m = \lambda$.

Theorem 7.11. *Let u be an eigenfunction to λ_1. Then $u > 0$ in Ω or $u < 0$ in Ω.*

"Proof". $u = u^+ + u^-$, where $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$. Then

$$m = Q(u) = \frac{\int_\Omega |\operatorname{div} u|^2 \, \mathrm{d}x}{\int_\Omega u^2 \, \mathrm{d}x} = \frac{\int_\Omega |\operatorname{div} u^+|^2 + \int_\Omega |\operatorname{div} u^-|^2}{\int_\Omega (u^+)^2 \, \mathrm{d}x + \int_\Omega (u^-)^2}.$$

(**Problem**: $u^-, u^+ \notin C^1(\Omega)$!)

$$\frac{\int_\Omega |\operatorname{div} u^\pm|^2}{\int_\Omega (u^\pm)^2 \, \mathrm{d}x} \geq m$$

implies $Q(u^\pm) = m$. Thus $u^\pm$ is solution to (7.1), with $\lambda = \lambda_1$. Therefore $u^+ > 0$ in Ω or $u^- < 0$ in Ω. □

Definition 7.12 (H_1)**.**

$$H_1 = \left\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : -\Delta u = \lambda_1 u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \right\}.$$

Theorem 7.13. *H_1 is 1-dimensional. That is, λ_1 is simple.*

Proof. Let $u, \tilde{u}$ be eigenfunctions to λ_1. We want to show that there exists c such that

$$u = c\tilde{u} \text{ in } \Omega.$$

Let

$$k = \frac{u(x_0)}{\tilde{u}(x_0)}$$

for some $x_0 \in \Omega$. Then

$$-\Delta \underbrace{(u - k\tilde{u})}_{=:w} = \lambda_1 \underbrace{(u - k\tilde{u})}_{=:w},$$

implying that $w > 0$ or $w < 0$ in Ω or $w \equiv 0$. Thus $w \equiv 0$. □

Theorem 7.14. *Let u and v be eigenfunctions to λ and μ respectively. Then either $\lambda = \mu$ or $\int_\Omega u \cdot v = 0$.*

Proof.

$$-\Delta u = \lambda u \Rightarrow \int_\Omega -\Delta uv \, \mathrm{d}x = \lambda \int_\Omega uv \, \mathrm{d}x$$
$$-\Delta v = \mu v \Rightarrow \int_\Omega -\Delta vu \, \mathrm{d}x = \mu \int_\Omega uv \, \mathrm{d}x.$$

But $\int_\Omega -\Delta uv \, \mathrm{d}x - \int_\Omega -\Delta vu \, \mathrm{d}x = \int_\Omega \nabla u \nabla v \, \mathrm{d}x = 0$, so

$$(\lambda - \mu) \int_\Omega uv \, \mathrm{d}x = 0.$$

□

How to find λ_2?

$$\lambda_1 = \inf \left\{ Q(w) : w \in C^2(\Omega) \cap C^2(\overline{\Omega}), w = 0 \text{ on } \partial\Omega, w \neq 0 \text{ in } \Omega \right\}$$

$$H_1 = \left\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : -\Delta u = \lambda_1 u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \right\}$$

$$\lambda_2 = \inf \left\{ Q(w) : w \in C^2(\Omega) \cap C^2(\overline{\Omega}), w = 0 \text{ on } \partial\Omega, w \not\equiv 0, \int_\Omega wu \, \mathrm{d}x = 0 \ \forall u \in H_1 \right\}$$

$$H_2 = \left\{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}) : -\Delta u = \lambda_2 u \text{ in } \Omega, u = 0 \text{ on } \partial\Omega \right\}$$

$$\lambda_3 = \inf \left\{ Q(w) : w \in C^2(\Omega) \cap C^2(\overline{\Omega}), w = 0 \text{ on } \partial\Omega, w \not\equiv 0, \int_\Omega wu \, \mathrm{d}x = 0 \ \forall u \in H_1 \cup H_2 \right\}$$

and so on...

Theorem 7.15 (Weyl Asymptotics)**.**

$$\lambda_k \approx \frac{4\pi^2 k^{\frac{2}{n}}}{(\alpha_n |\Omega|)^{\frac{2}{n}}}.$$

Figure 8: Two sets with same eigenvalues [3]

Theorem 7.16 (Polya Conjecture)**.** *Open:*

$$\lambda_k \geq \frac{4\pi^2 k^{\frac{2}{n}}}{(\alpha_n |\Omega|)^{\frac{2}{n}}}.$$

Best known:

$$\sum_{i=1}^{k} \lambda_i \geq \frac{n}{n+2} \frac{4\pi^2 k^{1+\frac{2}{n}}}{(\alpha_n |\Omega|)^{\frac{2}{n}}}.$$

Remark 7.17. Kac problem: Up to what extend the geometry of Ω can be recovered from $\{\lambda_i\}_{i=1}^{\infty}$?

Q: Are there any two different domains that have exactly same eigenvalues?

A: Yes.

Theorem 7.18. *Let $n = 2$, and Ω be smooth. Then*

$$\sum_{k=1}^{\infty} e^{\lambda_k t} \approx \frac{|\Omega|}{4\pi t} + \frac{|\partial\Omega|}{8\sqrt{\pi t}} + \frac{1}{6}(1-r) + O(t),$$

as $t \to 0$, where r is the number of holes in Ω.

8 Heat equation

Next we shall consider the *heat equation*: Find $u : \mathbb{R}^n \times]0, \infty[\to \mathbb{R}$, $(x,t) \mapsto u(x,t)$ such that

$$\partial_t u(x,t) - \underbrace{\Delta_x u(x,t)}_{\sum_{i=1}^{n} \partial_{x_i x_i} u(x,t)} = 0. \tag{8.1}$$

Also denoted $\partial_t u - \Delta u = 0$ or $u_t - \Delta u = 0$.

Derivation of equation: Let $u(x,t)$ be the temperature of something at point x at time t. By Fourier's law:

$$F = -k\nabla u,$$

where F is the *heat flux*, the rate of flow of heat (energy) per time through a unit volume of material and k is the *conductivity* of the material (at x, $k(x)$). Now, by Conservation of Enegy Law: Let $V \subset \mathbb{R}^n$ be a smooth set. Then

$$\begin{aligned}\int_V \partial_t u(x,t)\ \mathrm{d}x = \frac{\partial}{\partial_t}\int_V u(x,t)\ \mathrm{d}x &= -\int_{\partial V} F\cdot\nu\ \mathrm{d}S(x)\\ &= \int_{\partial V} k\nabla u\nu\ \mathrm{d}S(x)\\ &= \int_V \mathrm{div}(k\nabla u)\ \mathrm{d}x.\end{aligned}$$

Therefore $\partial_t u - \mathrm{div}(k\nabla u) = 0$. Put $k = 1$ and we have

$$\partial_t u - \Delta u = 0.$$

8.1 Fundamental solution

Definition 8.1 (Heat kernel Φ)**.** The function

$$\Phi(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}, & x\in\mathbb{R}^n, t>0\\ 0, & x\in\mathbb{R}^n, t\le 0.\end{cases}$$

is called the *fundamental solution* of heat equation, the *heat kernel.*

Lemma 8.2. *For all $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1.$$

Proof. For all $t > 0$, using the change of variables: $z = \frac{x}{\sqrt{4t}}$, $\mathrm{d}x = (\sqrt{4t})^n\ \mathrm{d}z$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\Phi(x,t)\ \mathrm{d}x &= \frac{1}{(4\pi t)^{\frac{1}{2}}}\int_{\mathbb{R}^n} e^{\frac{-|x|^2}{4t}}\ \mathrm{d}x\\ &= \frac{1}{n^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-|z|^2}\ \mathrm{d}z\\ &= \prod_{i=1}^n \underbrace{\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-z_i^2}\mathrm{d}z_i\right) = 1}_{Exercise}\end{aligned}$$

□

8.2 Adding boundary condition

To solve problem with a boundary condition consider

$$\begin{aligned}&\begin{cases}\partial_t u - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n\times]0,\infty[\\ u &= g \quad \text{on} \quad \mathbb{R}^n\times\{0\}\end{cases}\\ \Leftrightarrow\ &\begin{cases}\partial_t u(x,t) - \Delta u(x,t) &= 0 \quad \text{in} \quad \mathbb{R}^n\times]0,\infty[\\ u(x,0) &= g(x) \quad \text{on} \quad \mathbb{R}^n\times\{0\}.\end{cases}\end{aligned}$$

Then the *solution* is

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\ \mathrm{d}y \tag{8.2}$$

for all $x \in \mathbb{R}^n$ and $t > 0$.

Theorem 8.3. *Assume $g \in C(\mathbb{R}^n)$ is bounded. Define $u(x,t)$ as in (8.2). Then*

(i) $u \in C^\infty(\mathbb{R}^n \times]0,\infty[) \cap C(\mathbb{R}^n \times [0,\infty[)$

(ii) $\partial_t u - \Delta u = 0$ *in* $\mathbb{R}^n \times]0,\infty[$

(iii) $\lim_{\substack{(x,t)\to(x_0,t) \\ x\in\mathbb{R}^n \\ t>0}} u(x,t) = g(x_0)$ *for all* $x_0 \in \mathbb{R}^n$.

Proof. Proof of (i).

$$u(x,t) = \Phi(\cdot,t) * g(x)$$

$$\Phi(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|}{4t}} \in C^\infty(\mathbb{R}^n \times]0,\infty[$$

$\square_{\text{(i)}}$

Proof of (ii). We need following lemma. The proof is left as an exercise.

Lemma 8.4. *For all $t > 0$*

$$\partial_t \Phi(x,t) - \Delta_x \Phi(x,t) = 0.$$

By the lemma

$$\partial_t u(x,t) - \Delta u(x,t) = \int_{\mathbb{R}^n} \underbrace{(\partial_t \Phi(x-y,t) - \Delta\Phi_x \Phi(x-y,t))}_{=0} g(y)\ \mathrm{d}y \tag{8.3}$$

$$= 0. \tag{8.4}$$

$\square_{\text{(ii)}}$

Proof of (iii). TODO. $\square_{\text{(iii)}}$

$\square$

8.3 Non-homogenous heat equation

The non-homogenous heat equation is

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) &= f(x,t) \quad \text{in} \quad \mathbb{R}^n \times]0,\infty[\\ u(x,0) &= g(x) \quad \text{on} \quad \mathbb{R}^n \times \{0\}\,. \end{cases} \tag{8.5}$$

Theorem 8.5. *For all $x \in \mathbb{R}^n$ and $t \in]0,\infty[$,*

$$u(x,t) = \int_{\mathbb{R}^n} g(y)\Phi(x-y,t)\ dy + \int_0^t \int_{\mathbb{R}^n} f(y,s)\Phi(x-y,t-s)\ dy\ ds \tag{8.6}$$

is solution to (8.5).

Theorem 8.6. *Assume* $f \in C_0^{2,1}(\mathbb{R}^n \times [0,\infty[)$ *(that is,* $\partial_{xx} f$ *is continuous and* $\partial_t f$ *is countinuous) and that* f *has compact support in* $\mathbb{R}^n$ *at each* $t \in [0,\infty[$*. Define* $u \in \mathbb{R}^n \times]0,\infty[\to \mathbb{R}$ *as*

$$u(x,t) = \int_{\mathbb{R}^n} g(y)\Phi(x-y,t)\ dy + \int_0^t \int_{\mathbb{R}^n} f(y,s)\Phi(x-y,t-s)\ dy\ ds.$$

Then

(i) $u \in C^{2,1}(\mathbb{R}^n \times]0,\infty[)$

(ii) $\partial_t u - \Delta u = f$ *in* $\mathbb{R}^n \times]0,\infty[$

(iii) $\lim_{\substack{x \to x_0 \\ t \to 0^+}} u(x,t) = 0$ *for all* $x_0 \in \mathbb{R}^n$.

Proof. TODO □

8.4 Properties of solutions to heat equation

Definition 8.7 (Heat ball)**.** Fix $x \in \mathbb{R}^n$, $t > 0$ and $r > 0$. We define the *heat ball* by setting

$$E(x,t,r) = \left\{ (y,s) \in \mathbb{R}^{n+1} : \Phi(x-y,t-s) > \frac{1}{(4\pi)^{\frac{n}{2}} r^n} \right\},$$

where

$$\Phi(x-y,t-s) = \begin{cases} \dfrac{1}{(4\pi(t-s))^{\frac{n}{2}}} \cdot e^{\frac{|x-y|^2}{4(t-s)}}, & s < t \\ \qquad 0 & s \geq t \end{cases}$$

or by setting

$$E(x,t,r) = \left\{ (y,s) \in \mathbb{R}^{n+1} : t - r^{<} s < t, |x-y| < \mathbb{R}_r(s) \right\},$$

where

$$R_r(s) = \left(2n(t-s) \log \frac{r^2}{(t-s)} \right)^{\frac{1}{2}}.$$

Theorem 8.8 (Mean Value Property for solutions to heat equation)**.** *Let* $u \in C^{2,1}(\Omega \times]0,T[)$, $\Omega \subset \mathbb{R}^n$*. Denote* $\Omega_T = \Omega \times]0,T[$*. Assume* u *is a solution to the heat equation*

$$\partial_t - \Delta u = 0 \text{ in } \Omega_T.$$

Then

$$\begin{aligned} u(x,t) &= \frac{1}{2^{n+2}\pi^{\frac{n}{2}} r^n} \int \int_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2}\ dy\ ds \\ &= \frac{1}{2(4\pi)^{\frac{n}{2}} r^n} \int \int_{\partial E(x,t,r)} u(y,s) \frac{|y-x|}{t-s}\ dy\ ds, \end{aligned}$$

for all $E(x,t,r) \subset \Omega_T$.

Definition 8.9 (Parabolic boundary of Ω_T)**.** Let $\Omega_T = \Omega\times]0,T[$, where $T > 0$, $\Omega \subset \mathbb{R}^n$. We define the *parabolic boundary* of Ω_T by setting

$$\Gamma_T = (\Omega \times \{t = 0\}) \cap (\partial\Omega \times [0,T])\,.$$

Theorem 8.10 (Strong Maximum Principle)**.** *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation $\partial_t u - \Delta u = 0$ in Ω_T. Then*

(i) $\max\limits_{\overline{\Omega_T}} u = \max\limits_{\Gamma_T} u$

(ii) Suppose that $u(x_0, t_0) = \max\limits_{\overline{\Omega_T}} u$ for (x_0, t_0) in $\overline{\Omega_T} \setminus \Gamma_T$. Then

$$u(x,t) = u(x_0,t_0) \text{ in } \overline{\Omega} \times [0, t_0].$$

Proof. TODO □

Theorem 8.11 (Uniqueness)**.** *There exists at most one solution $u \in C^{2,1}(\Omega_T)\cap C(\overline{\Omega_T})$ of the initial-boundary value problem*

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{in} \quad \Omega_T \\ \qquad u &= g \quad \text{on} \quad \Gamma_T \end{cases}.$$

Theorem 8.12 (Mean Value Property for the Cauchy Problem)**.** *Suppose that $u \in C^{2,1}(\mathbb{R}^n\times]0,T[) \cap C(\mathbb{R}^n \times [0,T])$ solves the heat equation*

$$\begin{cases} \partial_t u - \Delta u &= 0 \quad \text{in} \quad \mathbb{R}^n\times]0,T[\\ \qquad u &= g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

and $u(x,t) \leq Ae^{a|x|^2}$, $x \in \mathbb{R}^n$, $0 \leq t \leq T$, $a, A > R^+$. Then

$$\sup_{\mathbb{R}^n\times[0,T]} u = \sup_{\mathbb{R}^n} g < \infty.$$

Proof. TODO □

Theorem 8.13 (Uniqueness)**.** *There exists at most one solution $u \in C^{2,1}(\mathbb{R}^n\times]0,T[)\cap C(\mathbb{R}^n \times [0,T])$ of*

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{in} \quad \mathbb{R}^n\times]0,T[\\ \qquad u &= g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\} \end{cases}$$

such that $u(x,t) \leq Ae^{a|x|^2}$, $x \in \mathbb{R}^n$, $0 \leq t \leq T$ for all $a, A \in \mathbb{R}$.

Theorem 8.14. *There exists at most one solution $u \in C^{2,1}(\mathbb{R}^n\times]0,T[)\cap C(\mathbb{R}^n\times[0,T])$ of*

$$\begin{cases} \partial_t u - \Delta u &= f \quad \text{in} \quad \Omega_T \\ \qquad u &= g \quad \text{on} \quad \Gamma_T \end{cases}.$$

Proof. TODO □

Theorem 8.15 (Backward Uniqueness)**.** *Suppose that $u, \tilde{u}$ are smooth solutions of*

$$\begin{cases} \partial_t u - \Delta u &= 0 \quad in \quad \Omega_T \\ \quad u &= 0 \quad on \quad \partial\Omega \times [0,T] \end{cases}$$

and

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} &= 0 \quad in \quad \Omega_T \\ \quad \tilde{u} &= 0 \quad on \quad \partial\Omega \times [0,T] \end{cases}.$$

If $u(x,T) = \tilde{u}(x,T)$ for all $x \in \Omega$, then $u = \tilde{u}$ in Ω_T.

Definition 8.16 (Parabolic Cylinder)**.** We define *parabolic cylinder* by setting

$$C(x,t,r) = \left\{(y,s) : |y-x| < r, t - r^2 < s < t\right\}.$$

Theorem 8.17. *Let $u \in C^{2,1}(\Omega \times]0,T[)$, $\Omega \subset \mathbb{R}^n$ be a solution to $\partial_t u - \Delta u = 0$ in Ω_T. Then $u \in C^\infty(\Omega_T)$ and we have the following estimates: For all $k, l \in \mathbb{N}$ there exists $C_{k,l,n} > 0$ depending only on k, l and n such that*

$$\max_{C(x_0,t_0,\frac{r}{2})} \left|D_x^k D_t^l u\right| \leq \frac{C_{k,l,n}}{r^{k+2l+n+2}} \int \int_{C(x_0,t_0,r)} |u(y,s)| \; dy \; ds$$

for all $C(x_0, t_0, r) \subset \Omega_T$.

Proof. TODO □

9 Wave equation

In this section we will consider the *wave equation*: Find $u : \Omega \times]0,\infty[\to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$, such that

$$\partial_{tt} u - \Delta_x u = 0. \tag{9.1}$$

Remark 9.1.

- $n = 1$: vibrating string
- $n = 2$: membrane
- $n = 3$: elastic solid

TODO: Derivation of equation.

9.1 Adding boundary condition

Consider wave equation with boundary condition

$$\begin{cases} \partial_{tt} u - \partial_{xx} u &= 0 \\ u(x,0) &= g(x) \\ \partial_t u(x,0) &= h(x). \end{cases} \tag{9.2}$$

Theorem 9.2. *Let $n = 1$. Then*

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(s)\ ds \quad \textit{(d'Alemhert's formula)}$$

is a solution to (9.2).

Proof. TODO □

Now, let $n \geq 2$ and consider wave equation with boundary condition

$$\begin{cases} \partial_{tt}u - \partial_{xx}u &= 0 \\ u(x,0) &= g(x) \\ \partial_t u(x,0) &= h(x). \end{cases} \tag{9.3}$$

Definition 9.3. Let us use the following notation. For all $x \in \mathbb{R}^n, t >, r > 0$, let

$$\begin{aligned} U(x,r,t) &= \fint_{\partial B(x,r)} u(y,t)\ \mathrm{d}S(y) \overset{r\to 0}{\to} u(x,t) = U(x,0,t) \\ G(x,r) &= \fint_{\partial B(x,r)} g(y)\ \mathrm{d}S(y) \overset{r\to 0}{\to} g(x) = G(x,0) \\ H(x,r) &= \fint_{\partial B(x,r)} h(y)\ \mathrm{d}S(y) \overset{r\to 0}{\to} h(x) = H(x,0) \end{aligned}$$

For $r < 0$,

$$\begin{aligned} U(x,r,t) &= U(x,-r,t) \\ G(x,r) &= G(x,-r) \\ H(x,r) &= H(x,-r). \end{aligned}$$

Theorem 9.4. *Fix $x \in \mathbb{R}^n$. Let u be a solution of (9.3). Then U solves the* Euler-Poisson-Darboux equation

$$\begin{cases} \partial_{tt}U - \partial_{rr}U &= \frac{n-1}{r}\partial_r U \quad \textit{in} \quad \mathbb{R}\times]0,\infty[\\ U(x,r,0) &= G(x,r) \quad \textit{for all} \quad r \in \mathbb{R} \\ \partial_t U(x,r,0) &= H(x,r) \quad \textit{for all} \quad r \in \mathbb{R}. \end{cases}$$

Proof.

$$\begin{aligned} \partial_r U(x,r,t) &= \partial_r \left(\fint_{\partial B(x,r)} u(y,t)\ \mathrm{d}S(y) \right) \\ &= \frac{r}{n} \fint_{B(x,r)} \underbrace{\Delta u(y,t)}_{=\partial_{tt}u(y,t)}\ \mathrm{d}S(y) \\ &= \frac{r}{n\alpha_n r^n} \int_{B(x,r)} \partial_{tt}u(y,t)\ \mathrm{d}y. \end{aligned}$$

$$
\begin{aligned}
\partial_r \left(r^{n-1} \partial_r U(x,r,t) \right) &= \frac{1}{n\alpha_n} \partial_r \left(\int_{B(0,r)} \partial_{tt} u(y,t) \; \mathrm{d}y \right) \\
&= \frac{1}{n\alpha_n} \int_{\partial B(x,r)} \partial_{tt} u(y,t) \; \mathrm{d}S(y) \\
&= \frac{r^{n-1}}{n\alpha_n r^{n-1}} \partial_t t \left(\int_{\partial B(x,r)} u(y,t) \; \mathrm{d}S(y) \right).
\end{aligned}
$$

Thus

$$
\partial_{rr} U(x,r,t) + \frac{n-1}{r} \partial_r U(x,r,t) = \partial_{tt} \left(⨍_{\partial B(x,r)} u(y,t) \; \mathrm{d}S(y) \right) = \partial_{tt} U(x,r,t).
$$

□

Let us try to solve (9.4). Let $n = 3$. Define

$$
\overline{U}(x,r,t) = rU(x,r,t).
$$

Then

$$
\begin{aligned}
\partial_r \partial_r \overline{U} &= \partial_r (r \partial_r U + U) = r \partial_{rr} U + 2 \partial_r U \\
\partial_{tt} \overline{U} &= r \partial_{tt} U.
\end{aligned}
$$

So

$$
\begin{cases}
\partial_{tt} \overline{U} - \partial_{rr} \overline{U} &= 0 \\
\overline{U}(r,0) &= rG(r) \\
\partial_t \overline{U}(r,0) &= rH(r).
\end{cases}
$$

Using d'Alembert's formula we obtain

$$
\overline{U}(r,t) = \frac{1}{2} \left((r+t) G(r+t) + (r-t) G(r-t) \right) + \frac{1}{2} \int_{r-t}^{r+t} sH(s) \; \mathrm{d}s. \tag{9.4}
$$

Bibliography

[1] DiBenedetto, E.: *Partial Differential Equations.*

[2] Evans, L. C.: *Partial Differential Equations.*

[3] Eskelinen, Matti: *PDE Lecture Notes, November 1st, 2012.*

[4] Folland, Gerald B.: *Introduction to Partial Differential Equations.*

[5] Gilbard, David & Trudinger, Neil S.: *Elliptic Equations of Second Order.*

[6] Hartikainen, Hans: *PDE Lecture Notes, October 5th, 2012.*

[7] Hartikainen, Hans: *PDE Lecture Notes, October 12th, 2012.*

[8] Strauss, W. A.: *Partial Differential Equations, An Introduction.*

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