Real analysis

lecture notes

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Contents

Introduction

These are the lecture notes of a real analysis course given in spring 2012. The first section is copied from Spyridon Dendrinos' notes and the latter three were typod during the lectures with the aid of Antti Luoto's notes. The notes are based on Measure Theory and Fine Properties of Functions by Evans and Gariepy [1].

About notation:

- X denotes a set and 2^X and $\mathcal{P}(X)$ the set of all subsets of X (or powerset of X).
- We denote the open interval from a to b with (a, b) and the closed interval with $[a, b]$.
- Set difference $A \setminus B = \{x \in A : x \notin B\}$ is denoted $A - B$.
- Note, that $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- For a real valued function f, $|f(x)|$ denotes the absolute value of $f(x)$ and for vector valued function f, $|f(x)|$ denotes the Euclidean norm $||f(x)||$.
- For sets A, B and point x,

$$\begin{aligned} \operatorname{dist}(x, A) &:= \inf_{y \in A} \{||x - y||\}, \\ \operatorname{dist}(A, B) &:= \inf_{\substack{x \in A, \\ y \in B}} \{||x - y||\} \text{ and} \\ \operatorname{diam} A &:= \sup_{x, y \in A} \{||x - y||\}. \end{aligned}$$

- Support of a function $f : X \to \mathbb{R}$ is the set

$$\operatorname{spt} f := \overline{\{x \in X : f(x) \neq 0\}}.$$

1 Review of measure and integration theory

Definition 1.1 (Outer measure)**.** An *outer measure* μ on a set X is a mapping $\mu : 2^X \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(A) \leq \mu(B)$, if $A \subset B \subset X$ and

(iii) $\mu\left(\cup_{k=1}^{\infty} A_k\right) \leq \sum_{k=1}^{\infty} \mu(A_k)$ for all sequences $(A_k)_{k=1}^{\infty}$ of subsets of X.

If $\mu(X) < \infty$, the outer measure is called *finite* .

Definition 1.2 (μ-measurable set)**.** Let μ be an outer measure on a set X. A subset A of X is said to be *μ-measurable* if

$$\mu(T) = \mu(T \cap A) + \mu(T \cap A^C),$$

for all $T \subset X$. Here $A^C = X - A = X \setminus A$. We may say set is *measurable* when we mean set is μ-measurable and μ is known in the context. Also, we may use abbreviation meas. for measurable.

Definition 1.3 (Measure restricted to a subset)**.** Let μ be an outer measure on X and $A \subset X$. Then μ *restricted to* A, $\mu\mid_A$, is the outer measure defined by

$$\mu|_A(B) = \mu(A \cap B)$$

for all $B \subset X$.

Theorem 1.4 (Properties of Measurable Sets)**.** *Let X be a set and μ be an outer measure on X.*

(i) Let $A \subset X$. If $\mu(A) = 0$, then A is measurable.

(ii) If $A \subset X$ is measurable, then A^C is also measurable.

(iii) If $A \subset X$, then any μ-measurable subset of A is also $\mu|_A$-measurable.

(iv) Let $(A_k)_{k=1}^{\infty}$ be a sequence of measurable sets in X. Then sets $\cup_{k=1}^{\infty} A_k$ and $\cap_{k=1}^{\infty} A_k$ are measurable.

(v) If the sets A_k are disjoint, then

$$\mu\left(\cup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k).$$

(vi) If $A_k \subset A_{k+1}$ for all $k \in \mathbb{N}$ then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\cup_{k=1}^{\infty} A_k\right).$$

(vii) If $A_k \supset A_{k+1}$ for all $k \in \mathbb{N}$ and $\mu(A_1) < \infty$, then

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\cap_{k=1}^{\infty} A_k\right).$$

Definition 1.5 (σ-algebra)**.** A collection of subsets $\mathcal{A} \subset \mathcal{P}(X)$ is called a *σ-algebra provided on X* if

(i) $\emptyset \in \mathcal{A}$,

(ii) $A \in \mathcal{A}$ implies $A^C \in \mathcal{A}$ and

(iii) $A_k \in \mathcal{A}$ for all $k \in \mathbb{N}$ implies $\cup_{k=1}^{\infty} A_k \in \mathcal{A}$.

Definition 1.6 (Measure space)**.** Given a σ-algebra $\mathcal{A} \subset \mathcal{P}(X)$, we define a *measure* on $\mathcal{P}(X)$ to be a mapping $\mu : \mathcal{A} \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$,

(ii) $\mu(\cup_{k=1}^{\infty} A_k) = \sum_{k=1}^{\infty} \mu(A_k)$ for all sequences $(a_k)_{k=1}^{\infty}$ of disjoint subsets of X.

The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*.

Remark 1.7. Given an outer measure on a set X, the collection of all μ-measurable subsets of X, call it $\mathcal{M}_\mu$, defines a σ-algebra, by Theorem 1.4 (iv). If we define $\tilde{\mu} : \mathcal{M}_\mu \to [0, \infty]$ by $\tilde{\mu}(A) = \mu(A)$ for any $A \in \mathcal{M}_\mu$, then μ is a measure. However, $\mathcal{M}_\mu$ may not be the largest σ-algebra of subset of X where this is possible to be done.

Definition 1.8 (σ-finite set)**.** Given an outer measure μ defined on a set X we call a subset $A \subset X$ *σ-finite set with respect to a measure* if we can write

$$A = \cup_{k=1}^{\infty} B_k,$$

where B_k are μ-measurable and $\mu(B_k) < \infty$ for all $k \in \mathbb{N}$. We may use abbreviation w.r.t. for with respect to.

Definition 1.9 (Borel σ-algebra, Borel set)**.** The *Borel σ-algebra of $\mathbb{R}^n$* is the smallest σ-algebra of $\mathbb{R}^n$ containing the open subsets of $\mathbb{R}^n$. Each element of the Borel σ-algebra is called a *Borel set*.

Remark 1.10 (Explanation of word "smallest").

- First of all, the powerset of $\mathbb{R}^n$ is a σ-algebra which contains all the open sets in $\mathbb{R}^n$.
- Any arbitrary intersection of a collection of σ-algebras is a σ-algebra .
  Check: Given family $\mathcal{A}_\alpha$ of σ-algebras suppose $B \in \cap_\alpha \mathcal{A}_\alpha$. Then

  $$B \in \mathcal{A}_\alpha \ \forall\alpha \Rightarrow B^C \in \mathcal{A}_\alpha \ \forall\alpha \Rightarrow B^C \in \cap_\alpha \mathcal{A}_\alpha.$$

  And if $B_k \in \cap_\alpha \mathcal{A}_\alpha \ \forall k \in \mathbb{N}$, then

  $$\begin{aligned} B_k \in \cap_\alpha \mathcal{A}_\alpha \ \forall k \in \mathbb{N} \ \forall\alpha \ &\Rightarrow \ \cup_{k=1}^{\infty} B_k \in \mathcal{A}_\alpha \ \forall\alpha \\ &\Rightarrow \ \cup_{k=1}^{\infty} B_k \in \cap_\alpha \mathcal{A}_\alpha. \end{aligned}$$

  So the smallest σ-algebra containing the open subsets of $\mathbb{R}^n$ is the intersection of all σ-algebras containing the open subsets of $\mathbb{R}^n$.

Definition 1.11 (Regular, Borel and Radon outer measures). (i) An outer measure μ on $\mathbb{R}^n$ is called *Borel* if every Borel set is μ-measurable .

(ii) An outer measure on X is called *regular* if for each set $A \subset X$ there exists a μ-measurable set B such that $A \subset B$ and $\mu(A) = \mu(B)$.

(iii) An outer measure μ on $\mathbb{R}^n$ is called *Borel regular* if μ is Borel and for each $A \subset \mathbb{R}^n$ there exists a Borel set B such that $A \subset B$ and $\mu(A) = \mu(B)$.

(iv) An outer measure μ on $\mathbb{R}^n$ is called a *Radon outer measure* if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^n$.

Note: There are alternative definitions of Radon outer measure elsewhere and Radon measure $\neq$ Radon outer measure.

Theorem 1.12. *Let μ be regular outer measure on X. If $A_k \subset A_{k+1}$ for all $k \in \mathbb{N}$, then*

$$\lim_{k\to\infty} \mu(A_k) = \mu\left(\cup_{k=1}^{\infty} A_k\right).$$

Note that here the A_k need not to be measurable.

Proof. Because μ is regular there exist measurable sets $(C_k)_{k=1}^{\infty}$ such that $A_k \subset C_k$ and $\mu(A_k) = \mu(C_k)$ for all $k \in \mathbb{N}$. Let

$$B_k := \cap_{j\geq k} C_j$$

so $B_1 \subset B_2 \subset \ldots$. The B_k are measurable and for each $j \geq k$

$$A_k \subset A_j \subset C_j$$

so

$$A_k \subset \cap_{j\geq k} C_j = B_k.$$

Thus $\mu(A_k) \leq \mu(B_k)$.
Also $\mu(A_k) = \mu(C_k) \geq \mu(\cap_{j\geq k} C_j) = \mu(B_k)$ hence

$$\mu(A_k) = \mu(B_k).$$

Then

$$\lim_{k\to\infty} \mu(A_k) = \lim_{k\to\infty} \mu(B_k) = \mu(\cup_{k=1}^{\infty} B_k) \geq \mu(\cup_{k=1}^{\infty} A_k).$$

But in addition

$$\begin{aligned} A_k \subset \cup_{j=1}^{\infty} A_j \quad &\Rightarrow \quad \mu(A_k) \leq \mu(\cup_{j=1}^{\infty} A_j) \\ &\Rightarrow \quad \lim_{k\to\infty} \mu(A_k) \leq \mu(\cup_{j=1}^{\infty} A_j), \end{aligned}$$

which ends the proof. □

Theorem 1.13. *Let μ be a Borel regular outer measure on $\mathbb{R}^n$. Suppose $A \subset \mathbb{R}^n$ is a Borel set. Then $\mu|_A$ is a Borel regular outer measure.*

Proof. First we need to show that $\mu|_A$ is a Borel outer measure. (i) Take any Borel set B and arbitrary T. Then

$$\begin{aligned}\mu|_A(B\cap T)+\mu|_A(T\cap B^C) &= \mu(A\cap B\cap T)+\mu(A\cap T\cap B^C)\\ &\overset{\text{regroup}}{=} \mu(A\cap T)=\mu|_A(T).\end{aligned}$$

(ii) Proof of the regularity. Take any $B\in\mathbb{R}^n$. Since μ is Borel regular there exists Borel set $D\subset\mathbb{R}^n$ such that $A\cap B\subset D$ and $\mu(A\cap B)=\mu(D)$. Because both A and D are Borel sets $A\cap D$ is Borel set and

$$\begin{aligned}A\cap D\supset A\cap B &\Rightarrow \mu(A\cap B)\le\mu(A\cap D)\text{ and}\\ A\cap D\subset D &\Rightarrow \mu(A\cap D)\le\mu(D)=\mu(A\cap B)\end{aligned}$$

thus

$$\mu(A\cap D)=\mu(A\cap B),$$

which is equal to

$$\mu|_A(D)=\mu|_A(B).$$

Choose $A^C\cup D$. That set is Borel and contains B and we also have

$$\mu|_A(A^C\cup D)=\mu|_A(D)=\mu|_A(B),$$

proving the theorem.

□

Theorem 1.14. *Let μ be a Borel regular outer measure on $\mathbb{R}^n$. Suppose $A\subset\mathbb{R}^n$ is μ-measurable and $\mu(A)<\infty$. Then $\mu|_A$ is a Radon outer measure.*

Proof. Let K be compact. Then $\mu|_A(K)=\mu(A\cap K)\le\mu(A)<\infty$. Take a Borel set B. For any arbitrary set $T\subset\mathbb{R}^n$

$$\mu|_A(T\cap B)+\mu|_A(T\cap B^C)=\mu(A\cap B\cap T)+\mu(A\cap B^C\cap T)=\mu(A\cap T)=\mu|_A(T)$$

so B is $\mu|_A$-measurable. Thus $\mu|_A$ is a Borel outer measure.

Remains to prove that $\mu|_A$ is Borel regular. Because μ is Borel regular there exists Borel set B such that $A\subset B$ and $\mu(A)=\mu(B)<\infty$. Since A is μ-measurable

$$\mu(B-A)=\mu(B)-\mu(A)=0.$$

Let $C\subset\mathbb{R}^n$.

$$\begin{aligned}\mu|_B(C) &= \mu(C\cap B)=\mu(C\cap B\cap A)+\mu\left(C\cap B\cap A^C\right)\\ &\le \mu(C\cap A)+\mu\left(B\cap A^C\right)\\ &= \mu(C\cap A)+0=\mu|_A(C).\end{aligned}$$

But since $A\cap C\subset B\cap C\Rightarrow\mu|_A(C)\le\mu|_B(C)$ we have

$$\mu|_B(C)=\mu|_A(C).$$

By Theorem 1.13 $\mu|_B$ is Borel regular, that is, for each $D\subset\mathbb{R}^n$ there exists Borel set E such that $\mu|_A(D)=\mu|_B(D)=\mu|_B(E)=\mu|_A(D)$, implying $\mu|_A$ is also Borel regular. □

Theorem 1.15. *Let μ be a Borel outer measure on $\mathbb{R}^n$ and let B be a Borel set.*

(i) If $\mu(B) < \infty$ then for each $\varepsilon > 0$ there exists closed set C such that $C \subset B$ and $\mu(B - C) < \varepsilon$.

(ii) Let μ be such that $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^n$. Then for each $\varepsilon > 0$ there exists an open set set U such that $B \subset U$ and $\mu(U - B) < \varepsilon$.

Proof. Proof of part (i). Since μ is Borel and $\mu(B) < \infty$, $\mu|_B$ is a finite Borel outer measure. Indeed, given any Borel set A and arbitrary $T \subset \mathbb{R}^n$

$$\begin{aligned} \mu|_B(A \cap T) + \mu|_B(A^C \cap T) &= \mu(B \cap A \cap T) + \mu(B \cap A^c \cap T) \\ &= \mu(B \cap T) = \mu|_B(T). \end{aligned}$$

Thus A is $\mu|_B$-measurable and $\mu|_B$ is a Borel outer measure. Let

$$\mathcal{F} := \{A \subset \mathbb{R}^n : A \text{ is } \mu\text{-meas. and } \forall \varepsilon > 0,\ \exists \text{ closed } C \subset A \text{ s.t. } \mu|_B(A - C) < \varepsilon\}.$$

Trivially, $\mathcal{F}$ contains all closed sets.
Claim 1: If $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then

$$A := \cap_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Proof of Claim 1. Fix $\varepsilon > 0$. Since $A_i \in \mathcal{F}$, there exists a closed set $C_i \subset A_i$ with $\mu|_B(A_i - C_i) < \varepsilon 2^{-i}$. Let

$$C := \cap_{i=1}^{\infty} C_i.$$

Then C is closed and

$$\begin{aligned} \mu|_B(A - C) &= \mu|_B\left(\cap_{i=1}^{\infty} A_i - \cap_{j=1}^{\infty} C_j\right) \\ &= \mu|_B\left(\cap_{i=1}^{\infty} A_i \cap \left(\cap_{j=1}^{\infty} C_j\right)^C\right) \\ &= \mu|_B\left(\cap_{i=1}^{\infty} A_i \cap \left(\cup_{j=1}^{\infty} C_j^C\right)\right) \\ &= \mu|_B\left(\cup_{j=1}^{\infty}\left(C_j^C \cap \left(\cap_{i=1}^{\infty} A_i\right)\right)\right) \\ &\le \mu|_B\left(\cup_{j=1}^{\infty}\left(C_j^C \cap A_j\right)\right) = \mu|_B\left(\cup_{j=1}^{\infty}(A_j - C_j)\right) \\ &\le \sum_{j=1}^{\infty} \underbrace{\mu|_B(A_j - C_j)}_{<\varepsilon 2^{-i}} < \varepsilon. \end{aligned}$$

Therefore $A \in \mathcal{F}$. $\square_{\text{Claim 1}}$

Claim 2: If $A_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $A := \cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Proof of Claim 2. Fix $\varepsilon > 0$ and given $i \in \mathbb{N}$. Since $A_i \in \mathcal{F}$, there exists a closed set C_i such that $C_i \subset A_i$ and $\mu|_B(A_i - C_i) < \varepsilon 2^{-i}$.
Each $A - \cup_{i=1}^{m} C_i = A \cap \left(\cup_{i=1}^{m} C_i\right)^C$ is $\mu|_B$-measurable. Then

$$\begin{aligned} \cap_{m=1}^{\infty}\left(A \cap \left(\cup_{i=1}^{m} C_i\right)^C\right) &= \cap_{m=1}^{\infty}\left(A \cap \left(\cap_{i=1}^{\infty} C_i^C\right)\right) \\ &= A \cap \left(\cap_{i=1}^{\infty} C_i^C\right) = A \cap \left(\cup_{i=1}^{\infty} C_i\right) \\ &= A - \left(\cup_{i=1}^{\infty} C_i\right). \end{aligned}$$

By Theorem 1.4 (vii), since $\mu|_B \left(\cup_{i=1}^{\infty} (A_i - C_i)\right) < \infty$,

$$\begin{aligned} \lim_{m\to\infty} \mu|_B \left(A - \cup_{i=1}^{m} C_i\right) &= \mu|_B \left(\cup_{i=1}^{\infty} A_i - \cup_{i=1}^{\infty} C_i\right) \\ &\leq \mu|_B \left(\cup_{i=1}^{\infty} (A_1 - C_i)\right) \\ &\leq \sum_{i=1}^{\infty} \mu|_B (A_1 - C_i) < \varepsilon. \end{aligned}$$

Therefore there exists $m \in \mathbb{N}$ such that $\mu|_B \left(A - \cup_{i=1}^{m} C_i\right) < \varepsilon$ and $\cup_{i=1}^{m} C_i$ is closed so $A \in \mathcal{F}$. $\square_{\text{Claim 2}}$

Since every open set can be written as a countable union of closed sets, Claim 2 shows that $\mathcal{F}$ contains all open sets. Now let

$$\mathcal{G} := \left\{A \in \mathcal{F} : A^C \in \mathcal{F}\right\},$$

which implies $A \in \mathcal{G} \Rightarrow A^C \in \mathcal{G}$. $\mathcal{G}$ contains all the open sets.

Claim 3: If $A_i \in \mathcal{G}$ for all $i \in \mathbb{N}$, then $A := \cup_{i=1}^{\infty} A_i \in \mathcal{G}$.

Proof of Claim 3. By Claim 2, $A \in \mathcal{F}$. Since $A_i^C \in \mathcal{F}$ for all $i \in \mathbb{N}$, by Claim 1

$$A^C = \left(\cup_{i=1}^{\infty} A_i\right)^C = \left(\cap_{i=1}^{\infty} A_i^C\right) \in \mathcal{F}.$$

Therefore $A \in \mathcal{G}$. $\square_{\text{Claim 3}}$

So $\mathcal{G}$ is a σ-algebra which contains all the open sets and therefore contains the Borel sets so in particular $B \in \mathcal{G}$ so $B \in \mathcal{F}$ implying that given any $\varepsilon > 0$ there exists closed set C such that $C \subset B$ and

$$\mu (B - C) = \mu \left(B \cap C^C\right) = \mu|_B \left(B \cap C^C\right) = \mu|_B (B - C) < \varepsilon$$

proving part (i). $\square_{\text{(i)}}$

Proof of part (ii). Denote by U_m the open ball of radius m, centre at 0. Then $U_m - B = U_m \cap B^C$ is a Borel set and $\mu\left((U_m - B)\right) < \mu (U_m) < \infty$. By part (i) we can find a closed set $C_m \subset U_m - B$ such that

$$\mu\left((U_m - B) - C_m\right) = \mu\left((U_m - C_m) - B\right) < \varepsilon 2^{-m}.$$

Let

$$U := \cup_{m=1}^{\infty} (U_m - C_m).$$

Then U is open. Now $C_m \subset B^C \Rightarrow B \subset C_m^C$ and therefore $U_m \cap B \subset U_m \cap C_m^C = U_m - C_m$ which implies

$$B = \cup_{m=1}^{\infty} (U_m \cap B) \subset \cup_{m=1}^{\infty} (U_m - C_m) = U.$$

And

$$\begin{aligned} \mu (U - B) &= \mu \left(\cup_{m=1}^{\infty} (U_m - C_m) \cap B^C\right) = \mu \left(\cup_{m=1}^{\infty} \left(U_m \cap C_m^C \cap B^C\right)\right) \\ &\leq \sum_{m=1}^{\infty} \mu\left((U_m - C) - B\right) < \varepsilon \sum_{m=1}^{\infty} 2^{-m} = \varepsilon. \end{aligned}$$

$\square_{\text{(ii)}}$

$\square$

Theorem 1.16. *Let μ be a Radon outer measure on $\mathbb{R}^n$. Then*

(i) for each set $A \subset \mathbb{R}^n$

$$\mu(A) = \inf \{\mu (U) : A \subset U, U \text{ open}\}$$

and

(ii) for each μ-measurable set $A \subset \mathbb{R}^n$

$$\mu (A) = \sup \{\mu (K) : K \subset A, K \text{ compact}\}.$$

Proof. Proof of part (i). Case $\mu (A) = \infty$: Nothing to prove.
Case $\mu (A) < \infty$: Assume first that A is a Borel set. Fix $\varepsilon > 0$. By Theorem 1.15 (ii), there exist an open set $U \supset A$ such that $\mu (U - A) < \varepsilon$. Then

$$\mu (U) = \mu (A) + \mu (U - A) < \mu (A) + \varepsilon.$$

$\therefore$ (i) holds for Borel sets.
Now, let A be arbitrary. Because μ is Borel regular, there exists Borel set B such that $A \subset B$ and $\mu (A) = \mu (B)$. Then

$$\begin{aligned} \mu (A) = \mu (B) &= \inf \{\mu (U) : B \subset U, U \text{ open}\} \\ &\geq \inf \{\mu (U) : A \subset U, U \text{ open}\}. \end{aligned}$$

Since $\mu (U) \geq \mu (A)$ for any $U \supset A$,

$$\mu (A) \leq \inf \{\mu (U) : A \subset U, U \text{ open}\}.$$

So $\mu (A) = \inf \{\mu (U) : A \subset U, U \text{ open}\}$. $\square_{\text{(i)}}$

Proof of part (ii). Assume A is μ-measurable and $\mu (A) < \infty$. By Theorem 1.14 $\mu|_A$ is Radon outer measure. Fix $\varepsilon > 0$. Applying (i) to $\mu|_A$ and A^C we obtain open set U with $A^C \subset U$ and

$$\mu|_A (U) = \mu (A \cap U) = \mu \left(U - A^C\right) < \varepsilon.$$

Let $C := U^C$. Then C is closed and $C \subset A$. Now

$$\mu|_A(U) = \mu|_A \left(C^C\right) = \mu \left(A \cap C^C\right) < \varepsilon$$

Therefore $\mu (C) \leq \mu (A)$ and $\mu (A) \leq \mu (C) + \varepsilon$. Thus

$$\mu (A) = \sup \{\mu (K) : K \subset A, K \text{ compact}\}.$$

Now consider case $\mu (A) = \infty$. Define annulus

$$D_k := \{x : k - 1 \leq |x| < k\}.$$

Then $\sum_{k=1}^{\infty} \mu(A \cap D_k) = \mu(A) = \infty$. Since μ is Radon outer measure we have that $\mu(D_k \cap A) < \infty$ for each $k \in \mathbb{N}$ so there exists closed set $C_k \subset D_k \cap A$ with $\mu(C_k) \geq \mu(D_k \cap A) - 2^{-k}$. Now $C_k \subset A$ for all $k \in \mathbb{N}$ so $\cup_{i=1}^{\infty} C_i \subset A$ and

$$\lim_{n\to\infty} \mu\left(\cup_{k=1}^{n} C_k\right) = \mu\left(\cup_{i=1}^{\infty} C_i\right) = \sum_{k=1}^{\infty} \mu(C_k) \geq \sum_{k=1}^{\infty} \left(\mu(D_k \cap A) - 2^{-k}\right) = \infty.$$

Since for each $n \in \mathbb{N}$, $\cup_{k=1}^{n} C_k$ is closed, we have

$$\mu(A) = \infty = \sup\{\mu(C) : C \subset A, C \text{ closed}\}.$$

Finally, let B_m be closed ball with centre 0 and radius m. Let C be closed and $C_m := C \cap B$. Then the C_m are compact and

$$\lim_{m\to\infty} \mu(C_m) = \mu(C)$$

and

$$\sup\{\mu(K) : K \subset A, K \text{ compact}\} = \sup\{\mu(C) : C \subset A, C \text{ closed}\}.$$

$\square_{\text{(ii)}}$

$\square$

Theorem 1.17 (Carathéodory's Criterion)**.** *Let μ be an outer measure on $\mathbb{R}^n$. If*

$$\mu(A \cup B) = \mu(A) + \mu(B)$$

for all sets $A, B \subset \mathbb{R}^n$ with $\operatorname{dist}(A, B) > 0$, then μ is a Borel outer measure.

Proof. We need to prove that every Borel set is μ-measurable. It suffices to show that every closed set is μ-measurable because from that follows that every open set is μ-measurable and therefore the open sets are contained in the σ-algebra of all μ-measurable sets. Since the Borel σ-algebra is contained in the σ-algebra of all μ-measurable sets, this means that every Borel set is μ-measurable.
Suppose $C \subset \mathbb{R}^n$ is closed. Given any $A \subset \mathbb{R}^n$ it is enough to show that

$$\mu(A) \geq \mu(A \cap C) + \mu(A - C).$$

This is obvious if $\mu(A) = \infty$ so assume $\mu(A) < \infty$. For all $n \in \mathbb{N}$ define

$$C_n := \left\{x \in \mathbb{R}^n : \operatorname{dist}(x, C) \leq \frac{1}{n}.\right\}$$

Then $\operatorname{dist}(A - C_n, A \cap C) \geq \frac{1}{n} > 0$. And therefore

$$\mu(A - C_n) + \mu(A \cap C) = \mu((A - C_n) \cup (A \cap C)) \leq \mu(A).$$

Claim 1:

$$\lim_{n\to\infty} \mu(A - C_n) = \mu(A - C).$$

Proof of the Claim 1. For all $k \in \mathbb{N}$, let

$$R_k := \left\{ x \in A : \frac{1}{k+1} < \operatorname{dist}(x, C) \le \frac{1}{k} \right\}.$$

Then $A - C = (A - C_n) \cup (\cup_{k=n}^{\infty} R_k)$ so that

$$\mu(A - C_n) \le \mu(A - C) \le \mu(A - C_n) + \sum_{k=n}^{\infty} \mu(R_k).$$

If $j \ge i + 2$ then $\operatorname{dist}(R_i, R_j) > 0$. It now follows that

$$\sum_{k=1}^{m} \mu(R_{2k}) = \mu(\cup_{k=1}^{m} R_{2k}) \le \mu(A)$$

and

$$\sum_{k=0}^{m} \mu(R_{2k+1}) = \mu(\cup_{k=0}^{m} R_{2k+1}) \le \mu(A).$$

Letting $m \to \infty$ we obtain

$$\sum_{k=1}^{\infty} \mu(R_k) \le 2\mu(A) < \infty.$$

Therefore

$$\begin{aligned} \lim_{n\to\infty} \mu(A - C_n) &\le \mu(A - C) \\ &\le \lim_{n\to\infty} \mu(A - C_n) + \underbrace{\lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(R_k)}_{\to 0 \text{ as } n \to \infty} \\ &= \lim_{n\to\infty} \mu(A - C_n). \end{aligned}$$

Which gives us the result

$$\lim_{n\to\infty} \mu(A - C_n) = \mu(A - C).$$

$\square_{\text{Claim 1}}$

Now by Claim 1,

$$\begin{aligned} \mu(A - C) + \mu(A \cap C) &= \lim_{n\to\infty} \mu(A - C_n) + \mu(A \cap C) \\ &\le \mu(A). \end{aligned}$$

Thus C is μ-measurable. $\square$

Definition 1.18 (μ-measurable function)**.** Let X be a set, Y be a topological space and μ be an outer measure on X. A function $F : X \to Y$ is called *μ-measurable* if for each open set $U \subset Y$, $f^{-1}(U)$ is μ-measurable.

Remark 1.19. Actually if $f : X \to Y$ is μ-measurable then $f^{-1}(B)$ is μ-measurable for every Borel set $B \subset Y$.

Definition 1.20 (σ-finite function)**.** Let X be a set and μ be an outer measure on X. A function $f : X \to [-\infty, \infty]$ is *σ-finite with respect to μ* if f is μ-measurable and $\{x : f(x) \neq 0\}$ is σ-finite with respect to μ.

Theorem 1.21 (Properties of Measurable Functions)**.** *(i) If $f, g : X \to \mathbb{R}$ are μ-measurable then so are $f + g$, fg, $|f|$, $\min(f, g)$, $\max(f, g)$ and f/g provided $g \neq 0$ on X.*

(ii) If the functions $f_k : X \to [-\infty, \infty]$ are μ-measurable for all $k \in \mathbb{N}$ then $\inf_{k\geq 1} f_k$, $\sup_{k\geq 1} f_k$, $\liminf_{k\to\infty} f_k$ and $\limsup_{k\to\infty} f_k$ are also μ-measurable.

Theorem 1.22. *Suppose $K \subset \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^n$ is continuous. Then there exists a continuous mapping $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that for all $x \in K$*

$$\tilde{f}(x) = f(x).$$

Proof. **Case $n = 1$:** For $x \in K^C$ and $s \in K$ set

$$u_s(x) := \max\left\{2 - \frac{|x - s|}{\operatorname{dist}(x, K)}, 0\right\}.$$

Notice $\frac{|x-s|}{\operatorname{dist}(x,K)} \geq 1$ so that $0 \leq u_s(x) \leq 1$. Also $u_s(x) = 0$ if $|x - s| \geq 2 \operatorname{dist}(x, K)$. Now $u_s(x)$ is continuous on K^C because it is a composition of continuous functions. Let us check this.
Claim 1: $\operatorname{dist}(x, K)$ is continuous on K^C.

Proof of Claim 1. Take a sequence $x_k \to x_0$. Since K is compact, for each x_k there exists $y_k \in K$ such that $\operatorname{dist}(x_k, K) = |x_k - y_k|$. Needs to show that

$$\operatorname{dist}(x_k, K) \to \operatorname{dist}(x_0, K).$$

Suppose $\operatorname{dist}(x_k, K) \not\to \operatorname{dist}(x_0, K)$, i.e. there exists $E > 0$ and a subsequence x_{k_e} such that

$$|\operatorname{dist}(x_{k_e}, K) - \operatorname{dist}(x_0, K)| = ||x_{k_e} - y_{k_e}| - \operatorname{dist}(x_0, K)| \geq E.$$

However, $y_{k_e} \in K$ and K is compact so there exists a subsequence $y_{k_{em}} \to y_0 \in K$ and that $x_{k_{em}}$. Now

$$\left|y_{k_{em}} - x_{k_{em}}\right| \to |x_0 - y_0|,$$

which is a contradiction, unless $\operatorname{dist}(x_0, K) \neq |y_0 - x_0|$.
We know that $\operatorname{dist}(x_0, K) \leq |y_0 - x_0|$. What if

$$\operatorname{dist}(x_0, K) = \inf_{y\in K} \{|x_0 - y|\} < |y_0 - x_0|? \tag{1.1}$$

If (1.1) holds there exists $\tilde{y} \in K$ and $E' > 0$ such that $|x_0 - \tilde{y}| < |x_0 - y_0| - E'$. Then

$$\begin{aligned}
\left|x_{k_{em}} - \tilde{y}\right| &\leq \left|x_{k_{em}} - x_0\right| + |x_0 - \tilde{y}| \\
&\leq \varepsilon + |x_0 - y_0| - E' \\
&\leq 2\varepsilon + \left|x_{k_{em}} - y_{k_{em}}\right| - E' \\
&= 2\varepsilon + \inf_{x \in K} \left\{\left|y - x_{k_{em}}\right|\right\} - E',
\end{aligned}$$

where $\varepsilon > 0$ can be chosen arbitrary small ($< E'/4$) for m sufficiently large. This is a contradiction and therefore,

$$|x_0 - y_0| = \operatorname{dist}(x_0, K).$$

$\therefore$ $\operatorname{dist}(x, K)$ is a continuous function. $\square_{\text{Claim 1}}$

Let $S := \{s_j \in K : j \in \mathbb{N}\}$ be a countable dense subset of K (e.g. the points in K with rational coordinates) and for all $x \in K^C$ define

$$\sigma(x) := \sum_{j=1}^{\infty} 2^{-j} u_{s_j}(x).$$

Then, for $x \in K^C$ we have $0 < \sigma(x) \leq 1$ because $\sigma(x) = 0$ would imply

$$|x - s_j| \geq 2 \operatorname{dist}(x, K)$$

for all s_j which is impossible since S is a dense subset of K. $\sigma(x)$ is continuous by the Weierstrass M-test since $\left|2^{-j} u_{s_j}(x)\right| < 2^{-j}$ and $\sum_{j=1}^{\infty} 2^{-j} < \infty$. Therefore $\sum_{j=1}^{\infty} 2^{-j} u_{s_j}(x)$ converges uniformely on K^C and thus $\sigma(x)$ is continuous on K^C.

Now, for all $x \in K^C$ and $k \in \mathbb{N}$, define

$$v_k(x) := \frac{2^{-k} u_{s_k}(x)}{\sigma(x)}.$$

Since

$$\sum_{k=1}^{\infty} \frac{2^{-k} u_{s_k}(x)}{\sigma(x)} \equiv 1$$

on K^C, the functions v_k form a partition of unity on K^C.

Finally, define

$$\tilde{f}(x) := \begin{cases} f(x) \text{ in } K \\ \sum_{k=1}^{\infty} v_k(x) f(s_k) \text{ in } K^C. \end{cases}$$

By the Weierstrass M-test $\tilde{f}$ is continuous on K^C. Let us check this.

Claim 2: $\tilde{f}$ is continuous on K^C.

Proof of Claim 2. Because f is continuous on a compact set K, f is bounded on K, say $|f(y)| \leq M$ on K. Then $|v_k(x) f(s_k)| < 2^{-k} M$ and therefore $\sum_{k=1}^{\infty} v_k(x) f(s_k)$ converges uniformely on K. $\square_{\text{Claim 2}}$

Now, let $a \in K$. Fix $\varepsilon > 0$. By the continuity of f and the compactness of K, there exists $\delta > 0$ such that

$$|f(a) - f(x)| < \varepsilon$$

for all $x \in K$ such that $|a - x| < \delta$. In particular, for all s_k such that $|s_k - a| < \delta$.

Suppose $x \in K^C$ and $|x - a| < \frac{\delta}{4}$. If $|a - s_k| \geq \delta$, then

$$\delta \leq |a - s_k| \leq |a - x| + |x - s_k| < \frac{\delta}{4} + |x - s_k|.$$

Therefore

$$|x - s_k| \geq \frac{3}{4}\delta > 2|x - a| \geq 2 \operatorname{dist}(x, K).$$

Thus $v_k(x) = 0$ whenever $|x - a| < \frac{\delta}{4}$ and $|a - s_k| \geq \delta$.

So if $x \in K^C$ and $|x - a| < \frac{\delta}{4}$, then

$$\begin{aligned} \left|\tilde{f}(x) - f(a)\right| &= \left|\sum_{k=1}^{\infty} v_k(x) f(s_k) - \sum_{k=1}^{\infty} v_k(x) f(a)\right| \\ &\leq \sum_{k=1}^{\infty} v_k(x) |f(s_k) - f(a)| < \varepsilon. \end{aligned}$$

For all $x \in K$ such that $|x - a| < \frac{\delta}{4}$ we also have

$$\left|\tilde{f}(x) - f(a)\right| = |f(x) - f(a)| < \varepsilon.$$

Case $n > 1$**:** Follows by extending each coordinate separately as in the case of $m = 1$. □

Theorem 1.23 (Lusin's Theorem)**.** *Let μ be a Borel regular outer measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ be μ-measurable. Assume $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$. Given $\varepsilon > 0$ there exists a compact set $K \subset A$ such that*

(i) $\mu(A - K) < \varepsilon$ and

(ii) $f|_K$ is continuous.

Proof. For each $i \in \mathbb{N}$, let $\{B_{ij}\}_{j=1}^{\infty} \subset \mathbb{R}^m$ be disjoint Borel sets such that $\mathbb{R}^m = \cup_{j=1}^{\infty} B_{ij}$ and $\operatorname{diam} B_{ij} < \frac{1}{i}$. Define

$$A_{ij} := A \cap f^{-1}(B_{ij}).$$

Then sets A_{ij} are disjoint. Since A and $f^{-1}(B_{ij})$ are μ-measurable so A_{ij} is also μ-measurable and

$$\begin{aligned} \cup_{j=1}^{\infty} A_{ij} &= \cup_{j=1}^{\infty} \left(A \cap f^{-1}(B_{ij})\right) = A \cap \left(\cup_{j=1}^{\infty} f^{-1}(B_{ij})\right) \\ &= A \cap \left(f^{-1}(\mathbb{R}^m)\right) = A \cap \mathbb{R}^n = A. \end{aligned}$$

By Theorem 1.14, $\mu|_A$ is Radon outer measure. Theorem 1.16 implies that for each A_{ij} there exists a compact set $K_{ij} \subset A_{ij}$ with

$$\mu|_A\left(A_{ij} - K_{ij}\right) < \varepsilon 2^{-i-j}.$$

Since A_{ij} are disjoint and $K_{ij} \subset A_{ij}$, set K_{ij} are also disjoint. Now,

$$\begin{aligned}
\mu\left(A - \cup_{j=1}^{\infty} K_{ij}\right) &= \mu|_A\left(A - \cup_{j=1}^{\infty} K_{ij}\right) \\
&= \mu|_A\left(\cup_{j=1}^{\infty} A_{ij} - \cup_{j=1}^{\infty} K_{ij}\right) \\
&\leq \mu|_A\left(\cup_{j=1}^{\infty}\left(A_{ij} - K_{ij}\right)\right) < \varepsilon 2^{-i}.
\end{aligned}$$

Because $A - \cup_{j=1}^{N} K_{ij} = A \cap \left(\cup_{j=1}^{N} K_{ij}\right)^C$ is μ-measurable and $\mu(A) < \infty$,

$$\lim_{N\to\infty} \mu\left(A - \cup_{j=1}^{N} K_{ij}\right) = \mu\left(A - \cup_{j=1}^{\infty} K_{ij}\right).$$

So there exists $N(i)$ such that $\mu\left(A - \cup_{j=1}^{N(i)} K_{ij}\right) < \varepsilon 2^{-i}$. Let

$$D_i := \cup_{j=1}^{N(i)} K_{ij}.$$

Then D_i is compact.

Given $i, j \in \mathbb{N}$, fix $b_{ij} \in B_{ij}$ and for all $x \in K_{ij} \subset D_i$, define $g_i : D_i \to \mathbb{R}^m$ by

$$g_i(x) = b_{ij}.$$

Since $K_{i1}, \ldots, K_{iN(i)}$ are compact and disjoint they are strictly positive distance apart so the g_i are continuous.

Furthermore, since $\operatorname{diam} B_{ij} < \frac{1}{i}$,

$$|f(x) - g_i(x)| < \frac{1}{i}$$

for all $x \in D_i$.

Let $K := \cap_{i=1}^{\infty} D_i$. Then K is compact and

$$\begin{aligned}
\mu(A - K) &= \mu\left(A \cap \left(\cap_{i=1}^{\infty} D_i\right)^C\right) = \mu\left(A \cap \left(\cup_{i=1}^{\infty} D_i^C\right)\right) \\
&= \mu\left(\cup_{i=1}^{\infty}\left(A \cap D_i^C\right)\right) \leq \sum_{i=1}^{\infty} \mu\left(A - D_i\right) \\
&< \varepsilon \sum_{i=1}^{\infty} 2^{-i} = \varepsilon.
\end{aligned}$$

Given $\delta > 0$, for all $x \in K$ we have

$$|f(x) - g_i(x)| < \frac{1}{i} < \delta,$$

when i is chosen large enough. So $g_i \to f$ uniformely on K and therefore $f|_K$ is continuous. □

Corollary 1.24. *Let μ be a Borel outer measure on $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}^m$ be μ-measurable. Assume $A \subset \mathbb{R}^n$ is μ-measurable and $\mu(A) < \infty$. Given $\varepsilon > 0$, there exists a continuous function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\mu\left(\left\{x \in A : \tilde{f}(x) \neq f(x)\right\}\right) < \varepsilon.$$

Proof. By Theorem 1.23 there exists a compact set $K \subset A$ such that $\mu(A - K) < \varepsilon$ and $f|_K$ is continuous. Then by Theorem 1.22 there exists a continuous function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\tilde{f}|_K = f|_K$$

and $\mu\left(\left\{x \in A : \tilde{f}(x) \neq f(x)\right\}\right) \leq \mu(A - K) < \varepsilon$. □

Theorem 1.25 (Egoroff's Theorem)**.** *Let μ be an outer measure on $\mathbb{R}^n$, for all $k \in \mathbb{N}$, $f_k : \mathbb{R}^n \to \mathbb{R}^m$ be μ-measurable, $A \subset \mathbb{R}^n$ be μ-measurable with $\mu(A) < \infty$ and $f_k \to g$ μ a.e. on A. (That is, except on a set of outer measure 0.) Then for each $\varepsilon > 0$ there exists a μ-measurable set $B \subset A$ such that*

(i) $\mu(A - B) < \varepsilon$ *and*

(ii) $f_k \to g$ *uniformely on B.*

Proof. For all $i, j \in \mathbb{N}$, define

$$C_{ij} := \cup_{k=j}^{\infty} \left\{x : |f_k(x) - g(x)| > 2^{-i}\right\}.$$

then $C_{ij+1} \subset C_{ij}$ for all $i, j \in \mathbb{N}$. Also, since $\mu(A) < \infty$,

$$\begin{aligned} \lim_{j\to\infty} \mu(A \cap C_{ij}) &= \mu\left(\cup_{j=1}^{\infty}(A \cap C_{ij})\right) \\ &= \mu\left(A \cap \left(\cap_{j=1}^{\infty} C_{ij}\right)\right) = 0, \end{aligned}$$

since for almost every $x \in A$ there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$|f_k(x) - g(x)| \leq 2^{-i}$$

or equivalently $x \notin C_{ik_0}$. So there exists $N(i) \in \mathbb{N}$ such that

$$\mu\left(A \cap C_{iN(i)}\right) < \varepsilon 2^{-i}.$$

Define

$$B := A - \cup_{i=1}^{\infty} C_{iN(i)}.$$

Then

$$\begin{aligned} \mu(A - B) &= \mu\left(A \cap \left(\cup_{i=1}^{\infty} C_{iN(i)}\right)\right) \\ &= \mu\left(\cup_{i=1}^{\infty}\left(A \cap C_{iN(i)}\right)\right) \\ &\leq \sum_{i=1}^{\infty} \mu\left(A \cap C_{iN(i)}\right) < \varepsilon. \end{aligned}$$

And given $i \in \mathbb{N}$, for all $k \geq N(i)$ and for all $x \in B$ we have

$$|f_k(x) - g(x)| \leq 2^{-i}.$$

In other words, $f_k \to g$ uniformely on B. □

Definition 1.26 (Simple function)**.** A *simple function on a set X* is a function that assumes only a finite number of values. We may use abbreviation s.f. for simple function.

Remark 1.27. Every simple function $f : X \to [-\infty, \infty]$ with range $f = \{a_1, a_2, \dots, a_k\}$ can be written in form

$$f = \sum_{i=1}^{k} a_i \chi_{A_i},$$

where the sets A_i are disjoint. Just choose $A_i = \{x \in X : f(x) = a_i\}$.

Theorem 1.28 (Approximation by Simple Functions)**.** *Let X be a set, μ be an outer measure on X and $f : X \to [0, \infty]$ be μ-measurable. Then there exists a sequence of finite-valued, μ-measurable, simple functions g_i defined on X such that*

$$g_1 \le g_2 \le g_3 \le \dots$$

and for all $x \in X$,

$$\lim_{i\to\infty} g_i(x) = f(x).$$

Furthermore, if f is σ-finite with respect to μ, the functions g_i can also be taken to be σ-finite.

Proof. For each $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$ such that $1 \le k \le n2^n$ let

$$A_{nk} := \left\{ x \in X : \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n} \right\}$$

and

$$B_n := \{x \in X : f(x) \ge n\}.$$

Define

$$g_n(x) := \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{A_nk}(x) + n\chi_{B_n}.$$

The sets A_{nk} and B_n are μ-measurable and so g_n are μ-measurable functions. Obviously for all $n \in \mathbb{N}$,

$$g_1 \le g_2 \le g_3 \le \dots \le g_n \le g_{n+1}$$

and for all $x \in B_n^C$,

$$|f(x) - g_n(x)| \le \frac{1}{2^n}.$$

Therefore for all $x \in X$,

$$\lim_{n\to\infty} g_n(x) = f(x).$$

Note that if f is σ-finite, all the A_{nk} and B_k defined above are also σ-finite, and hence the g_n are σ-finite. □

Definition 1.29 (Integral of a simple fuction with respect to a measure)**.** Given an outer measure μ on a set X and $g : X \to [0, \infty]$ a simple μ-measurable function. We define the *integral of the function g with respect to μ* by

$$\int g \, \mathrm{d}\mu = \sum_{0 \leq y \leq \infty} y \cdot \mu\left(g^{-1}(\{y\})\right).$$

And for any μ-measurable set A

$$\int_A g \, \mathrm{d}\mu = \sum_{0 \leq y \leq \infty} y \cdot \mu\left(g^{-1}(\{y\}) \cap A\right).$$

Definition 1.30 (Integral of a function)**.** Given an outer measure on a set X and a μ-measurable function $f : X \to [0, \infty]$. We define the *integral of the function f* by

$$\int f \, \mathrm{d}\mu = \sup\left\{\int g \, \mathrm{d}\mu : g : X \to [0, \infty] \text{ is } \mu\text{-meas. s.f. and } g \leq f \ \mu \text{ a.e.}\right\}.$$

And for any μ-measurable set A

$$\int_A f \, \mathrm{d}\mu = \sup\left\{\int_A g \, \mathrm{d}\mu : g : X \to [0, \infty] \text{ is } \mu\text{-meas. s.f. and } g \leq f \ \mu \text{ a.e. on } A\right\}.$$

Definition 1.31 (Integral of a function with respect to a measure)**.** Given an outer measure on a set X and a μ-measurable function $f : X \to [-\infty, \infty]$. Let $f^+ = \max\{f, 0\}$ and $f^- = \min\{-f, 0\}$ (then $f = f^+ - f^-$). Suppose either

$$\int f^+ \, \mathrm{d}\mu < \infty \text{ or } \int f^- \, \mathrm{d}\mu < \infty.$$

We then define the *integral of the function f with respect to a measure μ* by

$$\int f \, \mathrm{d}\mu = \int f^+ \, \mathrm{d}\mu - \int f^- \, \mathrm{d}\mu$$

and say that $\int f \, \mathrm{d}\mu$ *exists.*

Similarly, for any μ-measurable set $A \subset X$, if

$$\int_A f^+ \, \mathrm{d}\mu < \infty \text{ or } \int_A f^- \, \mathrm{d}\mu < \infty,$$

we define *integral of the fuction f over the set A with respect to the measure μ* by

$$\int_A f \, \mathrm{d}\mu = \int_A f^+ \, \mathrm{d}\mu - \int_A f^- \, \mathrm{d}\mu$$

and say $\int_A f \, \mathrm{d}\mu$ *exists.*

Definition 1.32 ($L^1(X,\mu)$, $L^1_{\text{loc}}(\mathbb{R}^n,\mu)$)**.** Given outer measure μ on a set X we define $L^1(X,\mu)$ to be the set of all μ-measurable functions f such that

$$\int |f| \, \mathrm{d}\mu < \infty$$

(i.e. $\int f^+ \, \mathrm{d}\mu < \infty$ and $\int f^- \, \mathrm{d}\mu < \infty$).

For $X = \mathbb{R}^n$ we define $L^1_{\text{loc}}(\mathbb{R}^n,\mu)$ to be the set of all μ-measurable functions f such that for all compact sets $K \subset \mathbb{R}^n$,

$$\int_K |f| \, \mathrm{d}\mu < \infty.$$

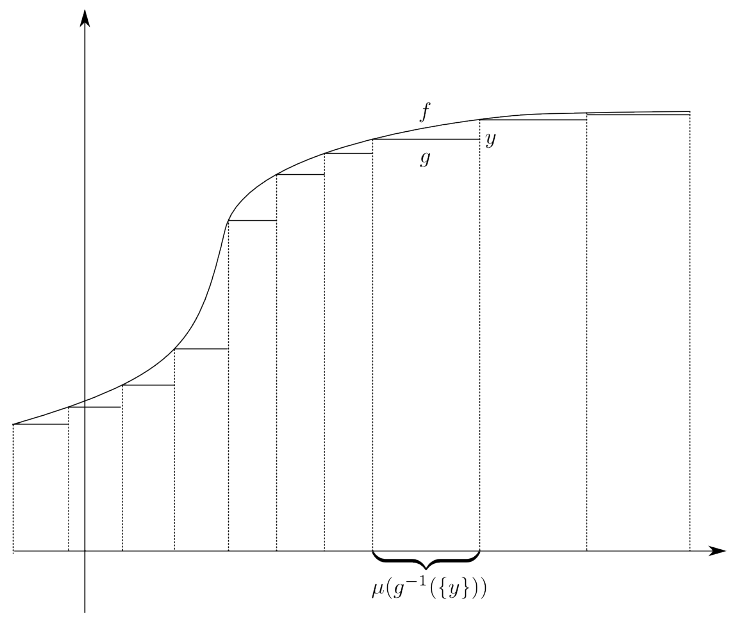

Figure 1: Approximation by a simple function

Note! From now on μ is an outer measure on a set X.

Theorem 1.33 (Fatou's Lemma). *Let $f_k : X \to [0,\infty]$ be μ-measurable for all $k \in \mathbb{N}$. Then*

$$\int \liminf_{k\to\infty} f_k \; d\mu \leq \liminf_{k\to\infty} \int f_k \; d\mu.$$

Proof. Let

$$g := \sum_{j=1}^{m} a_j \chi_{A_j}$$

be a non-negative μ-measurable simple function such that

$$g \leq \liminf_{k\to\infty} f_k \quad \mu \text{ a.e.}$$

with μ-measurable, disjoint $\{A_j\}_{j=1}^{\infty}$ and $a_j > 0$. (Note that, this is possible by Remark 1.27.) Fix $0 < t < 1$. For all $j, k \in \mathbb{N}$, define

$$B_{jk} := A_j \cap \{x \in X : f_l(x) > t a_j \text{ for all } l \geq k\}.$$

Then $\mu\left(A_j - \cup_{k=1}^{\infty} B_{jk}\right) = 0$ and for all $j, k \in \mathbb{N}$,

$$B_{jk} \subset B_{jk+1} \subset A_j.$$

So for all $m \in \mathbb{N}$

$$\begin{aligned} \int f_k \, \mathrm{d}\mu &\geq \sum_{j=1}^{m} \int_{A_j} f_k \, \mathrm{d}\mu \geq \sum_{j=1}^{m} \int_{B_{jk}} f_k \, \mathrm{d}\mu \\ &\geq t \sum_{j=1}^{m} a_j \mu\left(B_{jk}\right). \end{aligned}$$

Therefore, for all $0 < t < 1$ and simple μ-measurable $g \leq \liminf_{k\to\infty} f_k$,

$$\begin{aligned} \liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu &\geq t \sum_{j=1}^{m} a_j \liminf_{k\to\infty} \mu\left(B_{jk}\right) = t \sum_{j=1}^{m} a_j \lim_{k\to\infty} \mu\left(B_{jk}\right) \\ &= t \sum_{j=1}^{m} a_j \mu\left(A_j\right) = t \int g \, \mathrm{d}\mu. \end{aligned}$$

And therefore, by the definition of the integral of $\liminf_{k\to\infty} f_k$, we have

$$\liminf_{k\to\infty} \int f_k \, \mathrm{d}\mu \geq \int \liminf_{k\to\infty} f_k \, \mathrm{d}\mu.$$

□

Theorem 1.34 (Monotone Convergence Theorem). *For all $k \in \mathbb{N}$, let $f_k : X \to [0, \infty]$ be μ-measurable and*

$$f_k(x) \leq f_{k+1}(x) \quad \mu \text{ a.e. } x \in X$$

and $\lim_{k\to\infty} f_k$ exists μ a.e. $x \in X$. Then

$$\int \lim_{k\to\infty} f_k \ d\mu = \lim_{k\to\infty} \int f_k \ d\mu.$$

Proof. Since for almost every x and for all $j \in \mathbb{N}$

$$f_j(x) \leq \lim_{k\to\infty} f_k(x),$$

we have

$$\int f_j \ \mathrm{d}\mu \leq \int \lim_{k\to\infty} f_k \ \mathrm{d}\mu.$$

So

$$\lim_{j\to\infty} \int f_j \ \mathrm{d}\mu \leq \int \lim_{k\to\infty} f_k \ \mathrm{d}\mu.$$

By Fatou's Lemma we also have

$$\int \lim_{k\to\infty} f_k \ \mathrm{d}\mu \leq \liminf_{k\to\infty} \int f_k \ \mathrm{d}\mu = \lim_{k\to\infty} \int f_k \ \mathrm{d}\mu,$$

ending the proof. □

Theorem 1.35 (Dominated Convergence Theorem). *Let $g \in L^1(X, \mu)$ and f and $\{f_k\}_{k=1}^{\infty}$ be μ-measurable. Suppose $|f_k| \leq g$ and $f_k \to f$ μ a.e. as $k \to \infty$. Then*

$$\lim_{k\to\infty} \int |f_k - f| \ d\mu = 0.$$

Proof. Note that

$$\liminf_{k\to\infty} (2g - |f_k - f|) = 2g \quad \mu \text{ a.e.}$$

and

$$2g - |f - f_k| \geq 2g - |f| - |f_k| \geq 0 \quad \mu \text{ a.e.}.$$

So by Fatous Lemma,

$$\begin{aligned} \int 2g \ \mathrm{d}\mu &= \int \liminf_{k\to\infty} (2g - |f - f_k|) \ \mathrm{d}\mu \\ &\leq \liminf_{k\to\infty} \int 2g - |f - f_k| \ \mathrm{d}\mu \\ &= \int 2g - \limsup_{k\to\infty} \int |f - f_k| \ \mathrm{d}\mu. \end{aligned}$$

Thus we obtain

$$\limsup_{k\to\infty} \int |f_k - f| \ \mathrm{d}\mu \leq 0,$$

which proves the theorem. □

Theorem 1.36. *Suppose that* $g \in L^1(X,\mu)$ *and for all* $k \in \mathbb{N}$, $g_k \in L^1(X,\mu)$, f, f_k *are* μ*-measurable,* $|f_k| \le g_k$, $f_k \to f$ μ *a.e. ,* $g_k \to g$ μ *a.e. and*

$$\lim_{k\to\infty} \int g_k \ d\mu = \int g \ d\mu.$$

Then

$$\lim_{k\to\infty} \int |f_k - f| \ d\mu = 0.$$

Proof. Note that

$$\liminf_{k\to\infty} (2g_k - |f - f_k|) = 2g \quad \mu \text{ a.e.}$$

and

$$\liminf_{k\to\infty} (2g_k - |f - f_k|) \ge 0 \quad \mu \text{ a.e..}$$

By Fatous Lemma

$$\begin{aligned}
\int 2g \ \mathrm{d}\mu &= \liminf_{k\to\infty} (2g_k - |f - f_k|) \ \mathrm{d}\mu \\
&\le 2\liminf_{k\to\infty} \int g_k \ \mathrm{d}\mu - \limsup_{k\to\infty} \int |f - f_k| \ \mathrm{d}\mu \\
&= 2\int g \ \mathrm{d}\mu - \limsup_{k\to\infty} \int |f - f_k| \ \mathrm{d}\mu.
\end{aligned}$$

Therefore

$$\limsup_{k\to\infty} \int |f - f_k| \ \mathrm{d}\mu \le 0,$$

which is sufficient to prove the theorem. □

Theorem 1.37. *Assume that for all* $k \in \mathbb{N}$, $f, f_k \in L^1(X,\mu)$ *and*

$$\lim_{k\to\infty} \int |f - f_k| \ d\mu = 0.$$

Then there exists a subseqence $\left(f_{k_j}\right)_{j=1}^{\infty}$ *such that*

$$\lim_{j\to\infty} f_{k_j} = f \quad \mu \text{ a.e..}$$

Proof. Since $\lim_{k\to\infty} \int |f - f_k| \ \mathrm{d}\mu = 0$, given any $j \in \mathbb{N}$, we may find k_j such that

$$\int \left|f - f_{k_j}\right| \ \mathrm{d}\mu < 2^{-j}.$$

This implies that

$$\sum_{j=1}^{\infty} \int \left|f - f_{k_j}\right| \ \mathrm{d}\mu < \infty.$$

By the Monotone Convergence Theorem, since $\sum_{j=1}^{m} \left|f_{k_j} - f\right|$ is increasing as a function of m,

$$\int \sum_{j=1}^{\infty} \left|f - f_{k_j}\right| \, \mathrm{d}\mu \;=\; \sum_{j=1}^{\infty} \int \left|f - f_{k_j}\right| \, \mathrm{d}\mu < \infty$$

and so

$$\sum_{j=1}^{\infty} \left|f - f_{k_j}\right| < \infty \quad \mu \text{ a.e.}$$

which implies

$$\lim_{j\to\infty} \left|f - f_{k_j}\right| = 0 \quad \mu \text{ a.e.},$$

proving the theorem. □

Remark 1.38. Note that $\lim_{k\to\infty} \int |f - f_k| \, \mathrm{d}\mu = 0$ does **not** imply that $\lim_{k\to\infty} f_k = f$ μ a.e. .

Definition 1.39 (Product outer measure of two measures)**.** Let μ be an outer measure on a set X and ν be n outer mearuse on a set Y. For each $S \subset X \times Y$, we define the outer measure

$$\mu \times \nu : \mathcal{P}(X \times Y) \to [0, \infty]$$

by setting

$$(\mu \times \nu)(S) = \inf \left\{ \sum_{j=1}^{\infty} \mu(A_i) \, \nu(B_i) \right\},$$

where the infimum is taken over all collections of μ-measurable sets $A_i \subset X$ and ν-measurable sets $B_i \subset Y$, $i \in \mathbb{N}$, such that

$$S \subset \cup_{i=1}^{\infty} (A_i \times B_i).$$

The outer measure $\mu \times \nu$ is called the *product outer measure of* μ *and* ν.

Remark 1.40. Check subadditivity for product outer measure. For all $j \in \mathbb{N}$, let $S_j \in X \times Y$. Then

$$\begin{aligned}
(\mu \times \nu)\left(\cup_{j=1}^{\infty} S_j\right) &= \inf \left\{ \sum_{i=1}^{\infty} \mu(A_i) \, \nu(B_i) : \cup_{j=1}^{\infty} S_j \subset \cup_{i=1}^{\infty} A_i \times B_i \right\} \\
&\le \inf \left\{ \sum_{j,i=1}^{\infty} \mu(A_{ij}) \nu(B_{ij}) : \text{ for each } j \in \mathbb{N} \text{ s.t. } S_j \subset \cup_{i=1}^{\infty} A_{ij} \times B_{ij} \right\} \\
&\quad \text{and since } \mu(A_{ij}), \nu(B_{ij}) \ge 0, \\
&\le \sum_{j=1}^{\infty} \inf \left\{ \sum_{i=1}^{\infty} \mu(A_{ij}) \nu(B_{ij}) : S_j \subset \cup_{i=1}^{\infty} A_{ij} \times B_{ij} \right\} \\
&= \sum_{j=1}^{\infty} (\mu \times \nu)(S_j).
\end{aligned}$$

Theorem 1.41 (Fubini's Theorem)**.** *Let μ be an outer measure on X and ν be an outer measure on Y.*

(i) $\mu \times \nu$ is a regular outer measure on $X \times Y$ (even if μ and ν were not regular).

(ii) If $A \subset X$ is μ-measurable and $B \subset Y$ is ν-measurable, then $A \times B$ is $(\mu \times \nu)$-measurable and $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$.

(iii) If $S \subset X \times Y$ is σ-finite with respect to $\mu \times \nu$, then

$$S_y := \{x : (x, y) \in S\}$$

is μ-measurable for ν a.e. $y \in Y$,

$$S_x := \{y : (x, y) \in S\}$$

is ν-measurable for μ a.e. $x \in X$,

$$\int_Y \mu(S_y)\ d\nu(y) \text{ and } \int_X \nu(S_x)\ d\mu(x)$$

exists and are both equal to $(\mu \times \nu)(S)$.

(iv) If f is $(\mu \times \nu)$-measurable,

$$\int_{X \times Y} f\ d(\mu \times \nu)$$

exists and f is σ-finite with respect to $\mu \times \nu$ (in particular, if $f \in L^1(X \times Y, \mu \times \nu)$), then

$$\int_X f(x, y)\ d\mu(x) \text{ exists for } \nu \text{ a.e. } y,$$
$$\int_Y f(x, y)\ d\nu(y) \text{ exists for } \mu \text{ a.e. } x$$

and both

$$\int_Y \left(\int_X f(x, y)\ d\mu(x) \right)\ d\nu(y) \text{ and } \int_X \left(\int_Y f(x, y)\ d\nu(y) \right)\ d\mu(x)$$

exist and the last two integrals are both equal to

$$\int_{X \times Y} f\, d(\mu \times \nu).$$

Proof. Let $\mathcal{F}$ be the collection of all sets $S \subset X \times Y$ for which $\int_X \chi_S(x, y)\ \mathrm{d}\mu(x)$ exists ν a.e. $y \in Y$ (I.e. $\{x : (x, y) \in S\} = S_y$ is μ-measurable for ν a.e. y.) and

$$\int_Y \left(\int_X \chi_S(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y)$$

exists (I.e. in addition $\int_X \chi_S(x,y)\ \mathrm{d}\mu(x) = \mu(S_y)$ is ν-measurable.).

For $S \in \mathcal{F}$, define

$$\rho(S) := \int_Y \left(\int_X \chi_S\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y).$$

Let

$$\mathcal{P}_0 := \{A \times B : A \text{ is } \mu\text{-meas.}, B \text{ is } \nu\text{-meas.}\},$$
$$\mathcal{P}_1 := \left\{ \cup_{j=1}^{\infty} S_j : S_j \in \mathcal{P}_0 \right\}$$

and

$$\mathcal{P}_2 := \left\{ \cap_{j=1}^{\infty} S_j : S_j \in \mathcal{P}_1 \right\}.$$

Now we have $\mathcal{P}_0 \subset \mathcal{F}$, since

$$\begin{aligned} \int_X \chi_{A\times B}(x,y)\ \mathrm{d}\mu(x) &= \begin{cases} \mu(A) & \text{if } y \in B \\ 0 & \text{else} \end{cases} \\ &= \mu(A)\, \chi_B(y). \end{aligned}$$

and B is ν-measurable set, so $\mu(A)\, \chi_B(y)$ is a ν-measurable function.

If $A_1 \times B_1,\ A_2 \times B_2 \in \mathcal{P}_0$, then for intersection

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) \in \mathcal{P}_0$$

and for union

$$\begin{aligned} (A_1 \times B_1) \cup (A_2 \times B_2) &= (A_2 \times B_2) \cup ((A_1 \times B_1) - (A_2 \times B_2)) \\ &= (A_2 \times B_2) \cup ((A_1 - A_2) \times B_1) \cup ((A_1 \cap A_2) \times (B_1 - B_2)) \end{aligned}$$

Now

$$\begin{aligned} (A_2 \times B_2) \cap ((A_1 \cap A_2) \times (B_1 - B_2)) &= \emptyset, \\ (A_2 \times B_2) \cap ((A_1 \cap A_2) \times (B_1 - B_2)) &= \emptyset \text{ and} \\ ((A_1 - A_2) \times B_2) \cap ((A_1 \cap A_2) \times (B_1 - B_2)) &= \emptyset. \end{aligned}$$

i.e. $(A_1 \times B_1) \cup (A_2 \times B_2)$ is a disjoint union of three sets in $\mathcal{P}_0$ and further each member of $\mathcal{P}_1$ is a countable union of disjoint sets in $\mathcal{P}_0$. Therefore $\mathcal{P}_1 \subset \mathcal{F}$.

Claim 1: For all $S \subset X \times Y$,

$$(\mu \times \nu)(S) = \inf \{\rho(R) : S \subset R \in \mathcal{P}_1\}.$$

Proof of Claim 1. Every $R \in \mathcal{P}_1$ is of the form $R = \cup_{i=1}^{\infty} (A_i \times B_i)$. Then

$$\begin{aligned} \rho(R) &= \int_Y \left(\int_X \chi_R(x,y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) \text{ (well defined)} \\ &\le \sum_{i=1}^{\infty} \int_Y \left(\int_X \chi_{A_i \times B_i}(x,y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) \\ &= \sum_{i=1}^{\infty} \rho(A_i \times B_i) = \sum_{i=1}^{\infty} \mu(A_i)\nu(B_i), \end{aligned}$$

so

$$\inf\left\{\rho(R) : S \subset R \in \mathcal{P}_1\right\} \leq (\mu \times \nu)(S).$$

Moreover, for each $R \in \mathcal{P}_1$ there exists disjoint sequence $\left(A_j' \times B_j'\right)_{j=1}^{\infty} \subset \mathcal{P}_0$ such that $R = \cup_{j=1}^{\infty}\left(A_j' \times B_j'\right)$. Thus for all $S \subset R$,

$$\rho(R) = \sum_{j=1}^{\infty} \mu(A_j')\nu(B_j') \geq (\mu \times \nu)(S)$$

and therefore

$$\inf\left\{\rho(R) : S \subset R \in \mathcal{P}_1\right\} \geq (\mu \times \nu)(S),$$

proving Claim 1. $\square_{\text{Claim 1}}$

Proof of part (ii). Now, let $A \times B \in \mathcal{P}_0$. Then, for all $R \in \mathcal{P}_1$ such that $A \times B \subset R$,

$$(\mu \times \nu)(A \times B) \leq \mu(A)\,\nu(B) = \rho(A \times B) \leq \rho(R).$$

So by Claim 1,

$$(\mu \times \nu)(A \times B) = \inf\left(\rho(R) : A \times B \subset R \in \mathcal{P}_1\right),$$

and therefore we have

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B).$$

To prove that $A \times B$ is $(\mu \times \nu)$-measurable: Suppose $T \subset X \times Y$ and $T \subset R \in \mathcal{P}_1$, then also $R - (A \times B), R \cap (A \times B) \in \mathcal{P}_1$ and

$$(R - (A \times B)) \cap (R \cap (A \times B)) = \emptyset.$$

Therefore

$$\begin{aligned} & (\mu \times \nu)(T - (A \times B)) + (\mu \times \nu)(T \cap (A \times B)) \\ \leq\; & (\mu \times \nu)(R - (A \times B)) + (\mu \times \nu)(R \cap (A \times B)) \\ \leq\; & \rho(R - (A \times B)) + \rho(R \cap (A \times B)) = \rho(R). \end{aligned}$$

By Claim 1,

$$\begin{aligned} (\mu \times \nu)(T - (A \times B)) + (\mu \times \nu)(T \cap (A \times B)) &\leq \inf\left\{\rho(R) : T \subset R \in \mathcal{P}_1\right\} \\ &= (\mu \times \nu)(T). \end{aligned}$$

So $A \times B$ is $(\mu \times \nu)$-measurable.

$\square_{\text{(ii)}}$

Consider sequence of sets $R_j \in \mathcal{P}_1$. Let

$$R := \cap_{j=1}^{\infty} R_j.$$

Then $R \in \mathcal{P}_2$. We know that for all $j \in \mathbb{N}$, set $\{x : (x, y) \in R_j\}$ is μ-measurable for ν a.e. y. Now

$$\cap_{j=1}^{\infty} \{x : (x,y) \in R_j\} = \left\{x : (x,y) \in \cap_{j=1}^{\infty} R_j\right\}$$

is μ-measurable for ν a.e. y. And therefore

$$\int_X \chi_R(x,y) \, \mathrm{d}\mu(x)$$

exists for ν a.e. y.

Define sequence of functions

$$f_k(y) := \mu\left(\left\{x : (x,y) \in \cap_{i=1}^{k} R_j\right\}\right).$$

Each $R_j \in \mathcal{P}_1$ so $R_j = \cup_{i=1}^{\infty} A_{ij} \times B_{ij}$, where $A_{ij} \subset X$, $B_{ij} \subset Y$ and

$$\cap_{j=1}^{k} \left(\cup_{i=1}^{\infty} A_{ij} \times B_{ij}\right) \overset{\text{calculation}}{=} \cup_{1 \le i_n \le \infty} \left(\cap_{j=1}^{k} A_{i_j j} \times B_{i_j j}\right)$$

is a countable union of rectangles so it is in $\mathcal{P}_1 \subset \mathcal{F}$. Therefore each f_k is ν-measurable and $\lim_{k\to\infty} f_k$ is also ν-measurable.

$$\lim_{k\to\infty} f_k(y) = \lim_{k\to\infty} \mu\left(\left\{x : (x,y) \in \cap_{j=1}^{k} R_j\right\}\right) = \mu\left(\left\{x : (x,y) \in \cap_{j=1}^{\infty} R_j\right\}\right).$$

Therefore $R \in \mathcal{F}$.

Claim 2: For each $S \subset X \times Y$ there is a set $R \in \mathcal{P}_2$ such that $S \subset R$ and $\rho(R) = (\mu \times \nu)(S)$.

Proof of Claim 2. If $(\mu \times \nu)(S) = \infty$, define $R := X \times Y$. If $(\mu \times \nu)(S) < \infty$, then for each $j \in \mathbb{N}$, by Claim 1, there exists $R_j \in \mathcal{P}_1$ such that $S \subset R_j$ and

$$\rho(R_j) < (\mu \times \nu)(S) + \frac{1}{j}.$$

Define

$$R := \cap_{j=1}^{\infty} R_j \in \mathcal{P}_2.$$

Then $R \in \mathcal{F}$.

Also, $\rho\left(\cap_{j=1}^{k} R_j\right) \le \rho(R_k) < (\mu \times \nu)(S) + \frac{1}{k}$ so

$$\lim_{k\to\infty} \rho\left(\cap_{j=1}^{k} R_j\right) \le (\mu \times \nu)(S).$$

Therefore for all $j \in \mathbb{N}$, $S \subset R_j$, which implies $S \subset R$.

By the definition of $\mu \times \nu$, for all $k \in \mathbb{N}$, we have

$$(\mu \times \nu)(S) \le (\mu \times \nu)\left(\cap_{j=1}^{k} R_j\right) \le \rho\left(\cap_{j=1}^{k} R_j\right).$$

so

$$(\mu \times \nu)(S) \le \lim_{k\to\infty} \rho\left(\cap_{j=1}^{k} R_j\right).$$

Finally, by the Dominated Convergence Theorem,

$$\begin{aligned} \lim_{k\to\infty} \rho\left(\cap_{j=1}^{k} R_j\right) &= \rho(R) \\ \Rightarrow \rho(R) &= (\mu \times \nu)(S). \end{aligned}$$

$\square_{\text{Claim 2}}$

Proof of part (i). Every set in $\mathcal{P}_2$ is $(\mu \times \nu)$-measurable. (Recall that countable unions and intersections of measurable sets are also measurable.) So, given any $S \subset X \times Y$, by Claim 2 there exists $(\mu \times \nu)$-measurable $R \in \mathcal{P}_2$ such that

$$(\mu \times \nu)(S) = \rho(R) = (\mu \times \nu)(R).$$

Alternatively, by Dominated Convergence Theorem,

$$\rho(R) = \lim_{k\to\infty} \rho\left(\cap_{j=1}^{k} R_j\right) = \lim_{k\to\infty} (\mu \times \nu)\left(\cap_{j=1}^{k} R_j\right) = (\mu \times \nu)(R)$$

since R, R_j are all $(\mu \times \nu)$-measurable. $\square_{\text{(i)}}$

Proof of part (iii). Consider first $S \subset X \times Y$ with $(\mu \times \nu)(S) = 0$. Then there exists $R \in \mathcal{P}_2$ such that $S \subset R$ and

$$\rho(R) = \int_Y \left(\int_X \chi_R(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) = 0.$$

so for ν a.e.,

$$\int_X \chi_R(x, y)\ \mathrm{d}\mu(x) = 0,$$

thus

$$\mu(\{x : (x, y) \in R\}) = 0.$$

Since $S \subset R$, we have

$$\begin{aligned} & \{x : (x, y) \in S\} & \subset & \ \{x : (x, y) \in R\} \\ \Rightarrow\ & \mu(\{x : (x, y) \in S\}) & \leq & \ \mu(\{x : (x, y) \in R\}) = 0. \end{aligned}$$

Therefore $\{x : (x, y) \in S\}$ is μ-measurable for ν a.e. y and $\int_X \chi_S(x, y)\ \mathrm{d}\mu(x) = 0$ ν a.e. y and $\int_Y \left(\int_X \chi_S(x, y)\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\nu(y)$ exists and is equal to 0. So $S \in \mathcal{F}$ and $\rho(S) = 0$.

Next consider $(\mu \times \nu)$-measurable set $S \subset X \times Y$ with $(\mu \times \nu)(S) < \infty$. Then there exists $R \in \mathcal{P}_2$ such that $S \subset R$ and

$$(\mu \times \nu)(R) = \rho(R) = (\mu \times \nu)(S) \Rightarrow (\mu \times \nu)(R - S) = 0.$$

Hence by previous case we have

$$\begin{aligned} & \rho(R - S) & = & \ \int_Y \left(\int_X \chi_{R-S}(x, y)\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\nu(y) = 0 \\ \Rightarrow\ & \mu(\{x \in X : (x, y) \in R - S\}) & = & \ 0 \text{ for } \nu \text{ a.e. } y \end{aligned}$$

So $\{x \in X : (x, y) \in R - S\}$ is μ-measurable for ν a.e. y. Therefore

$$\{x \in X : (x, y) \in S\} = \{x \in X : (x, y) \in R\} - \{x \in X : (x, y) \in R - S\}$$

is μ-measurable for ν a.e. y and also $\mu(\{x : (x, y) \in S\}) = \mu(\{x : (x, y) \in R\})$ for ν a.e. y. Thus

$$\int_Y \mu(\{x : (x, y) \in S\})\ \mathrm{d}\nu(y)$$

exists and is equal to $\rho(R) = (\mu \times \nu)(S)$.

To end the proof of (iii), note the symmetry $x \longleftrightarrow y$. Case for σ-finite set $S \subset X \times Y$ follows by expressing such S as a disjoint countable union of $(\mu \times \nu)$-measurable sets of finite outer measure. $\square_{\text{(iii)}}$

Proof of part (iv). Reduces to (iii) when $f = \chi_S$, with set S σ-finite with respect to $\mu \times \nu$. Similarly for f, a non-negative, $(\mu \times \nu)$-measurable, σ-finite with respect to $\mu \times \nu$ simple function.

Now, let f be non-negative, $(\mu \times \nu)$-measurable, σ-finite with respect to $\mu \times \nu$. By Theorem 1.28, we can find an increasing sequence of non-negtive $(\mu \times \nu)$-measurable, σ-finite simple functions f_k such that $\lim_{k\to\infty} f_k = f$. Then for each $k \in \mathbb{N}$,

$$\int_X f_k(x, y)\ \mathrm{d}\mu(x)$$

is ν-measurable and by the Monotone Convergence Theorem

$$\int_X f(x, y)\ \mathrm{d}\mu(x) = \int_X \lim_{k\to\infty} f_k(x, y)\ \mathrm{d}\mu(x) = \lim_{k\to\infty} \int_X f_k(x, y)\ \mathrm{d}\mu(x)$$

is measurable. Similarly for $\int_Y f(x, y)\ \mathrm{d}\nu(y)$.

In addition, again by the Monotone Convergence Theorem,

$$\begin{aligned}
\int_Y \left(\int_X f(x, y\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) &= \int_Y \left(\int_X \lim_{k\to\infty} f_k(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) \\
&= \int_Y \left(\lim_{k\to\infty} \int_X f_k(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) \\
&= \lim_{k\to\infty} \int_Y \left(\int_X f_k(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) \\
&= \lim_{k\to\infty} \int_{X\times Y} f_k(x, y)\ \mathrm{d}(\mu \times \nu)(x, y) \\
&= \int_{X\times Y} \lim_{k\to\infty} f_k(x, y)\ \mathrm{d}(\mu \times \nu)(x, y) \\
&= \int_{X\times Y} f\ \mathrm{d}(\mu \times \nu).
\end{aligned}$$

Similarly for $\int_X \left(\int_Y f(x, y\ \mathrm{d}\nu(y) \right)\ \mathrm{d}\mu(x)$, giving us the equality.

For general σ-finite and $(\mu \times \nu)$-measurable function $f : X \times Y \to [-\infty, \infty]$

$$\int_{X\times Y} f\ \mathrm{d}(\mu \times \nu) \text{ exists } \Rightarrow \int_{X\times Y} f^+\ \mathrm{d}(\mu \times \nu) < \infty \text{ or } \int_{X\times Y} f^-\ \mathrm{d}(\mu \times \nu) < \infty.$$

Suppose without loss of generality that $\int_{X\times Y} f^+\ \mathrm{d}(\mu \times \nu) < \infty$. Then

$$\int_{X\times Y} f^+\ \mathrm{d}(\mu \times \nu) = \int_Y \left(\int_X f^+(x, y)\ \mathrm{d}\mu(x) \right)\ \mathrm{d}\nu(y) < \infty.$$

So $\int_X f^+\ \mathrm{d}\mu(x) < \infty$ for ν a.e. y and therefore $\int_X f(x, y)\ \mathrm{d}\mu(x)$ exists for ν a.e. y and $\int_X f(x, y)\ \mathrm{d}\mu(x) = \int_X f^+(x, y)\ \mathrm{d}\mu(x)$. Thus we have

$$\int_Y \left(\int_X f(x, y)\ \mathrm{d}\mu(x) \right)^+\ \mathrm{d}\nu(y) < \infty$$

and so $\int_Y \left(\int_X f(x,y)\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\nu(y)$ exists.

$$\begin{aligned}\int_{X\times Y} f\ \mathrm{d}(\mu\times\nu) &= \int_{X\times Y} f^+\ \mathrm{d}(\mu\times\nu) - \int_{X\times Y} f^-\ \mathrm{d}(\mu\times\nu)\\ &= \int_Y\int_X f^+(x,y)\ \mathrm{d}\mu(x)\ \mathrm{d}\nu(y) - \int_Y\int_X f^-(x,y)\ \mathrm{d}\mu(x)\ \mathrm{d}\nu(y)\\ &= \int_Y \left(\int_X f(x,y)\ \mathrm{d}\mu(x)\right)\ \mathrm{d}\nu(y).\end{aligned}$$

Similarly for $\int_X \left(\int_Y f(x,y)\ \mathrm{d}\nu(y)\right)\ \mathrm{d}\mu(x)$. $\square_{\text{(iv)}}$

$\square$

Definition 1.42 (Lebesgue outer measure)**.** The one-dimensional *Lebesgue outer measure* $\mathcal{L}^1$ on $\mathbb{R}$ is defined by

$$\mathcal{L}^1(A) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam} C_i : A \subset \cup_{i=1}^{\infty} C_i, C_i \subset R\right\}$$

for any subset $A \subset \mathbb{R}$.

Definition 1.43 (n-dimensional Lebesgue outer measure)**.** We inductively define the *n-dimensional Lebesgue outer measure* $\mathcal{L}^n$ on $\mathbb{R}^n$ by

$$\mathcal{L}^n := \mathcal{L}^{n-1} \times \mathcal{L}^1 = \underbrace{\mathcal{L}^1 \times \cdots \times \mathcal{L}^1}_{n \text{ times}}.$$

2 Covering and differentiation theorems

Definition 2.1 (Cover of a set, fine cover)**.** (i) A collection $\mathcal{F}$ of closed balls in $\mathbb{R}^n$ is called a *cover of a set* $A \subset \mathbb{R}^n$ if

$$A \subset \bigcup_{B \in \mathcal{F}} B.$$

(ii) A cover $\mathcal{F}$ of a set $A \subset \mathbb{R}^n$ is called a *fine cover* of B if, in addition, for every $x \in A$,

$$\inf \{\operatorname{diam} B : x \in B, B \in \mathcal{F}\} = 0.$$

Theorem 2.2 (Vitali's Covering Theorem[1])**.** *Let $\mathcal{F}$ be any collection of closed balls in $\mathbb{R}^n$ with nonempty interior and with*

$$\sup \{\operatorname{diam} B : B \in \mathcal{F}\} < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} \tilde{B}$$

where $\tilde{B}$ is the closed ball which is concentric with B and has 5 times the radius of B. (For a ball $B = B(x, r)$, we denote $\tilde{B} := B(x, 5r)$.)

Proof. Basic observation: If B and B' are closed balls such that $B \cap B' \neq \emptyset$ and $\operatorname{diam} B' \leq 2 \operatorname{diam} B$, then $B' \subset \tilde{B}$.

We pick the countable family of $\mathcal{G}$ in the following way:
Let

$$D := \sup \{\operatorname{diam} B : B \in \mathcal{F}\} < \infty.$$

Define

$$\mathcal{F}_j := \left\{ B \in \mathcal{F} : \frac{D}{2^j} < \operatorname{diam} B \leq \frac{D}{2^{j-1}}, j \in \mathbb{N} \right\}.$$

Define $\mathcal{G}_j$ inductively as follows:

- Let $\mathcal{G}_1$ be any maximal disjoint subcollection of balls in $\mathcal{F}_1$. Maximal means that after having picked $\mathcal{G}_1$, there does not exist any other disjoint ball $B' \in \mathcal{F}_1$ such that B' is disjoint from all balls $B \in \mathcal{G}_1$ (otherwise we should have added it to $\mathcal{G}_1$).

Note there are at most countably many balls in $\mathcal{G}_1$ because each B has nonempty interior and therefore contains a point with rational coordinates. So, if there were uncountably many balls in $\mathcal{G}_1$ we would have uncountably many rationals.

- Inductive step: Assuming $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_{k-1}$ have been selected, we choose $\mathcal{G}_k$ to be any maximal disjoint subcollection of the set

$$\left\{ B \in \mathcal{F}_k : B \cap B' = \emptyset \text{ for all } B' \in \cup_{j=1}^{k-1} \mathcal{G}_j \right\}.$$

[1] Also known as "5-ball Theorem" or "Vitali Covering Lemma".

Note the reasoning as above, there are countably many balls in each $\mathcal{G}_k$ for all $k \in \mathbb{N}$.

Define

$$\mathcal{G} := \cup_{j=1}^{\infty} \mathcal{G}_j.$$

Then $\mathcal{G}$ also contains countably many disjoint balls.

Now fix some ball $B \subset \mathcal{F}$. Then there exists $j \in \mathbb{N}$ such that $B \in \mathcal{F}_j$. Since $\mathcal{G}_j$ is maximal, there exists a ball $B' \in \cup_{k=1}^{j} \mathcal{G}_k \subset \mathcal{G}$ such that $B \cap B \neq \emptyset$.

$$\begin{aligned} \operatorname{diam} B' &> \frac{D}{2^j} \quad \text{and} \quad \operatorname{diam} B \leq \frac{D}{2^{j-1}} \\ \Rightarrow \operatorname{diam} B &\leq \frac{D}{2^{j-1}} < 2 \operatorname{diam} B'. \end{aligned}$$

Therefore, by the basic observation $B \subset \tilde{B}'$. □

Corollary 2.3. *Assume that $\mathcal{F}$ is a fine cover of A by closed balls with nonempty interior and*

$$\sup \{\operatorname{diam} B : B \in \mathcal{F}\} < \infty.$$

Then there exists a countable family $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that for each finite subset $\{B_1, \ldots, B_m\} \subset \mathcal{F}$ we have

$$A - \cup_{k=1}^{m} B_k \subset \cup_{B \in \mathcal{G} - \{B_1, \ldots, B_m\}} \tilde{B},$$

where $\tilde{B}$ are as in Theorem 2.2.

Proof. Let $\mathcal{G}$ be as in the proof of Theorem 2.2. Let $\{B_1, \ldots, B_m\} \subset \mathcal{F}$. If $A \subset \cup_{k=1}^{m} B_k$ we have nothing to prove. Otherwise, let $x \in A - \cup_{k=1}^{m} B_k$. Since the balls in $\mathcal{F}$ are closed and $\mathcal{F}$ is a fine cover, there exists $B \in \mathcal{F}$ with $x \in B$ and $B \cap B_k = \emptyset$ for all $k \in \{1, \ldots, m\}$.

Now

$$\operatorname{dist}(x, \cup_{i=1}^{m} B_k) > 0.$$

(Otherwise there would exist a sequence $(x_k) \subset \cup_{i=1}^{m} B_k$ such that $x_k \to x$ implying $x \in \cup_{k=1}^{m} B_k$.) But then, as in the proof of Theorem 2.2, there exists a ball $B' \in \mathcal{G} - \{B_1, \ldots, B_m\}$ such that $B \cap B' \neq \emptyset$ and therefore $B \subset \tilde{B}'$.

□

Corollary 2.4. [2] *Let $U \subset \mathbb{R}^n$ be open and $\delta > 0$. There exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that* $\operatorname{diam} B \leq \delta$ *for all $B \in \mathcal{G}$ and*

$$\mathcal{L}^n (U - \cup_{B \in \mathcal{G}} B) = 0.$$

Proof. Fix $\theta \in \left(1 - \frac{1}{5^n}, 1\right)$ and assume first that

$$\mathcal{L}^n(U) < \infty.$$

Claim: There exists a finite collection $\{B_i\}_{i=1}^{M_1}$ of disjoint closed balls in U such that $\operatorname{diam} B_i < \delta$ for all $1 \leq i \leq M_1$ and

$$\mathcal{L}^n \left(U - \cup_{i=1}^{M_1} B_i\right) \leq \theta \mathcal{L}^n(U).$$

[2] Also known as "Vitali Covering Theorem".

Proof of Claim. Let

$$\mathcal{F}_1 := \{B : B \subset U \text{ closed ball with nonempty interior with } \operatorname{diam} B < \delta\}.$$

Since U is open, $\mathcal{F}_1$ covers U.

By Theorem 2.2, there exists a countable disjoint family $\mathcal{G}_1 \subset \mathcal{F}$ such that $U \subset \cup_{B\in\mathcal{F}_1} B \subset \cup_{B\in\mathcal{G}_1} \tilde{B}$. Then

$$\begin{aligned}\mathcal{L}^n(U) &\leq \sum_{B\in\mathcal{G}_1} \mathcal{L}^n(\tilde{B}) = 5^n \sum_{B\in\mathcal{G}_1} \mathcal{L}^n(B) \\ &= 5^n \mathcal{L}^n(\cup_{B\in\mathcal{G}_1} B).\end{aligned}$$

Therefore $\mathcal{L}^n(\cup_{B\in\mathcal{G}_1} B) \geq \frac{1}{5^n}\mathcal{L}^n(U)$ and we obtain

$$\mathcal{L}^n(U - \cup_{B\in\mathcal{G}_1} B) \leq \mathcal{L}^n(U) - \mathcal{L}^n(\cup_{B\in\mathcal{G}_1} B) \leq \left(1 - \frac{1}{5^n}\right)\mathcal{L}^n(U).$$

Since $\mathcal{G}_1$ is countable, we may enumerate the $B \in \mathcal{G}_1$ in a sequence $B_1, B_2, \ldots$. Then

$$\lim_{m\to\infty} \mathcal{L}^n(U - \cup_{i=1}^m B_i) = \mathcal{L}^n(U - \cup_{B\in\mathcal{G}_1} B) \leq \left(1 - \frac{1}{5^n}\right)\mathcal{L}^n(U).$$

So there exists $M_1 \in \mathbb{N}$ such that $\mathcal{L}^n\left(U - \cup_{i=1}^{M_1} B_i\right) \leq \theta\mathcal{L}^n(U)$. $\square_{\text{Claim}}$

Now let

$$U_2 := U - \cup_{i=1}^{M_1} B_i$$

and

$$\mathcal{F}_2 := \{B : B \subset U_2 \text{ with } \operatorname{diam} B < \delta\}$$

and apply the above claim in order to find finitely many disjoint balls $B_{M_1+1}, \ldots, B_{M_2}$ in $\mathcal{F}_1$ such that

$$\begin{aligned}\mathcal{L}^n\left(U - \cup_{i=1}^{M_2} B_i\right) &= \mathcal{L}^n\left(U_2 - \cup_{i=M_1+1}^{M_2} B_i\right) \\ &\leq \theta\mathcal{L}^n(U_2) \leq \theta^2\mathcal{L}^n(U).\end{aligned}$$

We may continue in this manner and obtain a countable collection of disjoint balls such that for all $k \in \mathbb{N}$

$$\mathcal{L}^n\left(U - \cup_{i=1}^{M_k} B_i\right) \leq \theta^k\mathcal{L}^n(U).$$

Since $\theta^k \to 0$ as $k \to \infty$, the corollary follows in the case $\mathcal{L}^n(U) < \infty$.

If $\mathcal{L}^n(U) = \infty$, we split U in a disjoint (modulo a set of measure 0) union. For all $m \in \mathbb{N}$, let

$$U_m := \{x \in U : m - 1 < |x| < m\}.$$

Then $\mathcal{L}^n(U - \cup_{m=1}^\infty U_m) = 0$ and $\mathcal{L}^n(U_m) < \infty$. Apply the above to U_m. For each m there exists $\mathcal{G}_m$ (countable collection of disjoint balls in U_m) such that

$$\mathcal{L}^n(U_m - \cup_{B\in\mathcal{G}_m} B = 0.)$$

Let $\mathcal{G} := \cup_{m=1}^\infty \mathcal{G}_m$ and we obtain $\mathcal{L}^n(U - \cup_{B\in\mathcal{G}} B) = 0$, ending the proof. $\square$

Theorem 2.5 (Besicovitch's Covering Theorem)**.** *There exists a constant N_n, depending only on the dimension n, with the following property: If $\mathcal{F}$ is any collection of closed balls in $\mathbb{R}^n$ with nonempty interior and with*

$$\sup\{\operatorname{diam} B : B \in \mathcal{F}\} < \infty$$

and if A is the set of centres of balls in $\mathcal{F}$, then there exist $\mathcal{G}_1, \ldots, \mathcal{G}_{N_n} \subset \mathcal{F}$ such that each $\mathcal{G}_i$, $i \in \{1, \ldots, N_n\}$ is a countable collection of disjoint balls in $\mathcal{F}$ and

$$A \subset \cup_{i=1}^{N_n} \left(\cup_{B \in \mathcal{G}_i} B\right).$$

Proof. To see the proof, please refer to Mattila [3, p. 30-34]. For the best constant N_n and the idea of the proof of the Theorem 2.5, see [2]. □

Corollary 2.6. *Let μ be a Borel regular outer measure on $\mathbb{R}^n$ and $\mathcal{F}$ be any collection of closed balls with nonempty interior. Let A denote the set of centres of the balls in $\mathcal{F}$. Assume $\mu(A) < \infty$ and for each $a \in A$*

$$\inf\{r : B(a, r) \in \mathcal{F}\} = 0.$$

Then for each open set $U \subset \mathbb{R}^n$ there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\cup_{B \in \mathcal{G}} B \subset U$$

and

$$\mu\left((A \cap U) - \cup_{B \in \mathcal{G}} B\right) = 0.$$

Proof. Fix $1 - \frac{1}{N_n} < \theta < 1$.
Claim: There exists a finite collection $\{B_1, \ldots, B_{M_1}\}$ of disjoint closed balls in U such that

$$\mu\left((A \cap U) - \cup_{i=1}^{M_1} B_i\right) \le \theta \mu(A \cap U).$$

Proof of Claim. Let

$$\mathcal{F}_1 := \{B : B \in \mathcal{F}, \operatorname{diam} B \le 1, B \subset U\}.$$

By Theorem 2.5, there exist families $\mathcal{G}_1, \ldots, \mathcal{G}_{N_n}$ of disjoint alls in $\mathcal{F}_1$ such that

$$A \cap U \subset \cup_{i=1}^{N_n} \left(\cup_{B \in \mathcal{G}_i} B\right).$$

Thus

$$\mu(A \cap U) \le \sum_{i=1}^{N_n} \mu\left(A \cap U \cap \left(\cup_{B \in \mathcal{G}_i} B\right)\right).$$

Therefore, by the pigeonhole principle, there exists $j \in \{1, \ldots, N_n\}$ such that

$$\mu\left(A \cap U \cap \left(\cup_{B \in \mathcal{G}_i} B\right)\right) \ge \frac{1}{N_n} \mu(A \cap U).$$

By Theorem 2.2 there exists balls $B_1, \ldots, B_{M_1} \in \mathcal{G}_j$ such that

$$\mu(\underbrace{A \cap U \cap (\cup_{i=1}^{M_1} B_i)}_{\text{Not necessarily } \mu\text{-meas.!}}) \geq (1-\theta)\mu\,(A \cap U)\,.$$

But also

$$\mu\,(A \cap U) = \mu\left(A \cap U \cap \left(\cup_{i=1}^{M_1} B_i\right)\right) + \mu\left(A \cap U - \left(\cup_{i=1}^{M_1} B_i\right)\right),$$

since $\cup_{i=1}^{M_1} B_i$ is μ-measurable and therefore

$$\begin{aligned} \mu\left(A \cap U - \left(\cup_{i=1}^{M_1} B_i\right)\right) &= \mu\,(A \cap U) - \mu\left(A \cap U \cap \left(\cup_{i=1}^{M_1} B_i\right)\right) \\ &\leq \theta\mu\,(A \cap U)\,. \end{aligned}$$

$\square_{\text{Claim}}$

Now let

$$U_2 := U - \cup_{i=1}^{M_1} B_i\ ,\ \mathcal{F}_2 := \{B : B \in \mathcal{F}, \operatorname{diam} B \leq 1, B \subset U_2\}$$

and as in Claim, find finitely many disjoint balls $B_{M_1+1}, \ldots, B_{M_2}$ in $\mathcal{F}_2$ such that

$$\begin{aligned} \mu\left((A \cap U) - \cup_{i=1}^{M_2} B_i\right) &= \mu\left((A \cap U) - \left(\cup_{i=M_1+1}^{M_2} B_i\right)\right) \\ &= \theta\mu\,((A \cap U_2) \leq \theta^2\mu\,(A \cap U)\,. \end{aligned}$$

Continue this process to obtain a countable collection of disjoint balls in $\mathcal{F}$ and inside U such that

$$\mu\left((A \cap U) - \left(\cup_{i=1}^{M_k} B_i\right)\right) \leq \theta^k\mu\,(A \cap U)\,.$$

Since $\mu\,(A) < \infty$ and $\theta^k \to 0$, letting $k \to \infty$ we obtain the theorem.

$\square$

Definition 2.7. Let μ and ν be Radon outer measures on $\mathbb{R}^n$. For each point $x \in \mathbb{R}^n$, we define

$$\overline{D}_\mu\nu(x) := \begin{cases} \limsup_{r\to\infty} \dfrac{\nu\,(B(x,r))}{\mu\,(B(x,r))}, & \text{if } \mu\,(B(x,r)) > 0 \text{ for all } r > 0 \\ +\infty, & \text{if } \mu\,(B(x,r)) = 0 \text{ for some } r > 0. \end{cases}$$

$$\underline{D}_\mu\nu(x) := \begin{cases} \liminf_{r\to\infty} \dfrac{\nu\,(B(x,r))}{\mu\,(B(x,r))}, & \text{if } \mu\,(B(x,r)) > 0 \text{ for all } r > 0 \\ +\infty, & \text{if } \mu\,(B(x,r)) = 0 \text{ for some } r > 0. \end{cases}$$

Definition 2.8. If $\overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x) < +\infty$, we say that ν is *differentiable with respect to μ at x* and write

$$D_\mu\nu(x) = \overline{D}_\mu\nu(x) = \underline{D}_\mu\nu(x).$$

We say $D_\mu\nu$ is the *derivative of ν with respect to μ*. We also call $D_\mu\nu$ the *density of ν with respect to μ*.

Theorem 2.9. *Let μ and ν be Radon outer measures on $\mathbb{R}^n$. Let $0 < \alpha < \infty$. Then*

(i) $A \subset \left\{x \in \mathbb{R}^n : \underline{D}_\mu \nu(x) \leq \alpha\right\}$ *implies* $\nu(A) \leq \alpha\mu\,(A)$.

(ii) $A \subset \left\{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\right\}$ *implies* $\nu(A) \geq \alpha\mu\,(A)$.

Proof. Proof of part (i). For part (i) assume first that $\mu\,(\mathbb{R}^n)\,, \nu\,(\mathbb{R}^n) < \infty$. Fix $\varepsilon > 0$. Let U be open, $A \subset U$. Define

$$\mathcal{F} := \{B := B(a,r) : a \in A, B \subset U, \nu\,(B) \leq (\alpha+\varepsilon)\mu\,(B)\}\,.$$

Then, since for each $a \in A$,

$$\liminf_{r\to 0} \frac{\mu\,(B(a,r))}{\nu\,(B(a,r))} \leq \alpha$$

there exists sequence (r_k), converging to 0 such that

$$\lim_{k\to\infty} \frac{\mu\,(B(a,r_k))}{\nu\,(B(a,r_k))} \leq \alpha.$$

So given any $\delta > 0$, there exists $k \in \mathbb{N}$ such that $r_k \leq \delta$ and $\frac{\mu(B(a,r_k))}{\nu(B(a,r_k))} \leq \alpha+\varepsilon$ implying $B(a,r_k) \in \mathcal{F}$ and therefore

$$\inf\{r : B(a,r) \in \mathcal{F}\} = 0$$

for each $a \in A$.

Now, by Corollary 2.6, there exists a countable collection $\mathcal{G}$ of disjoint balls in $\mathcal{F}$ such that

$$\nu\,((A\cap U) - \cup_{B\in\mathcal{G}}B) = \nu\,(A - \cup_{B\in G}B) = 0.$$

Then

$$\begin{aligned}\nu\,(A) &\leq \nu\,(\cup_{B\in\mathcal{G}}B) + \nu\,(A - \cup_{B\in\mathcal{G}}B) \leq \sum_{B\in\mathcal{G}}\nu\,(B)\\ &\leq (\alpha+\varepsilon)\sum_{B\in\mathcal{G}}\mu\,(B) \leq (\alpha+\varepsilon)\mu\,(U)\,,\end{aligned}$$

where last inequality follows from $\cup_{B\in\mathcal{G}}B \subset U$ and from $\mathcal{G}$ consisting of disjoint balls. This is true for all open sets $U \supset A$. Since $\mu\,(A) = \inf\{\mu(U) : A \subset U \text{ open}\}$, $\nu(A) \leq (\alpha+\varepsilon)\mu\,(A)$ for all $\varepsilon > 0$ so

$$\nu(A) \leq \alpha\mu(A).$$

Generally (not assuming $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$), consider

$$\nu_k := \nu|_{B(0,k)}\ ,\ \mu_k := \mu|_{B(0,k)}.$$

Then $\mu_k(\mathbb{R}^n) < \infty$ and $\nu_k(\mathbb{R}^n) < \infty$. Now

$$A \subset \left\{x \in \mathbb{R}^n : \underline{D}_{\mu_k}\nu_k(x) \leq \alpha\right\}.$$

Therefore

$$\begin{aligned} \nu_k \left(A \cap \operatorname{int} B(0,k)\right) &\leq \alpha \mu_k \left(A \cap \operatorname{int} B(0,k)\right) \\ \Rightarrow \nu \left(A \cap \operatorname{int} B(0,k)\right) &\leq \alpha \mu \left(A \cap \operatorname{int} B(0,k)\right). \end{aligned}$$

Letting $k \to \infty$ we obtain $\nu(A) \leq \alpha\mu(A)$. (The limit exists, even though A may not be μ, ν-measurable, since ν and μ are regular.) $\Box_{\text{(i)}}$

Proof of part (ii). For part (ii) consider

$$A \subset \left\{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\right\}.$$

Since

$$\left\{x \in \mathbb{R}^n : \overline{D}_\mu \nu(x) \geq \alpha\right\} = \left\{x \in \mathbb{R}^n : \underline{D}_\nu \mu(x) \leq \frac{1}{\alpha}\right\},$$

we obtain $\mu(A) \leq \frac{1}{\alpha}\nu(A)$ and therefore $\nu(A) \geq \alpha\mu(A)$. $\Box_{\text{(ii)}}$

$\Box$

Theorem 2.10. *Let μ and ν be Radon outer measures on $\mathbb{R}^n$. Then $D_\mu \nu$ exists and is finite μ a.e. Furthermore, $D_\mu \nu$ is μ-measurable.*

Proof. **Claim 1:** $D_\mu \nu$ exists and is finite μ a.e.

Proof of Claim 1. Let

$$I := \left\{x : \overline{D}_\mu \nu(x) = +\infty\right\}$$

and for $0 < a < b$ let

$$R(a,b) := \left\{x : \underline{D}_\mu \nu(x) < a < b < \overline{D}_\mu \nu(x) < +\infty\right\}.$$

For each $\alpha > 0$, $I \subset \left\{x : \overline{D}_\mu \nu(x) \geq \alpha\right\}$. Then by Theorem 2.9, $\nu(I) \geq \alpha\mu(I)$ implying

$$\mu(I) \leq \frac{1}{\alpha}\nu(I).$$

Letting $\alpha \to \infty$, we obtain $\mu(I) = 0$, so $\overline{D}_\mu \nu$ is finite μ a.e.

Also by Theorem 2.9 we have

$$b\mu\left(R(a,b)\right) \leq \nu\left(R(a,b)\right) \leq a\mu\left(R(a,b)\right)$$

and since $b > a$, we have $\mu(R(a,b)) = 0$.

Moreover,

$$\left\{x : \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x) < +\infty\right\} = \bigcup_{\substack{0<a<b \\ a,b\in\mathbb{Q}}} R(a,b),$$

implying

$$\mu\left(\left\{x : \underline{D}_\mu \nu(x) < \overline{D}_\mu \nu(x) < +\infty\right\}\right) = 0,$$

so $D_\mu \nu$ exists and is finite μ a.e. $\Box_{\text{Claim 1}}$

Claim 2: For each $x \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} \limsup_{y \to x} \mu\left(B(y,r)\right) &\leq \mu\left(B(x,r)\right) \\ \limsup_{y \to x} \nu\left(B(y,r)\right) &\leq \nu\left(B(x,r)\right). \end{aligned}$$

Proof of Claim 2. Choose sequence $y_k \subset \mathbb{R}^n$ with $y_k \to x$. Let

$$f_k := \chi_{B(y_k,r)}\ ,\ f := \chi_{B(x,r)}.$$

Then $\limsup_{k\to\infty} f_k \leq f$ and by Fatou's lemma,

$$\liminf_{k\to\infty}(1-f_k) \geq (1-f).$$

For μ we obtain

$$\begin{aligned} \int_{B(x,2r)} (1-f)\ \mathrm{d}\mu &\leq \int_{B(x,2r)} \liminf_{k\to\infty}(1-f_k)\ \mathrm{d}\mu \\ &\leq \liminf_{k\to\infty} \int_{B(x,2r)} (1-f_k)\ \mathrm{d}\mu. \end{aligned}$$

Therefore $\mu\left(B(x,2r)\right) - \mu\left(B(x,r)\right) \leq \liminf_{k\to\infty}\left(\mu\left(B(x,2r)\right) - \mu\left(B(y_k,r)\right)\right)$, thus

$$\limsup_{k\to\infty} \mu\left(B(y_k,r)\right) \leq \mu\left(B(x,r)\right).$$

For ν we obtain the claim similarly. $\square_{\text{Claim 2}}$

Claim 3: $D_\mu \nu$ is μ-measurable.

Proof of Claim 3. By Claim 2, for all $r > 0$, the functions $x \mapsto \mu\left(B(x,r)\right)$ and $x \mapsto \nu\left(B(x,r)\right)$ are upper semicontinuous which implies that the sets $\{x : \mu\left(B(x,r)\right) < \alpha\}$ and $\{x : \nu\left(B(x,r)\right) < \alpha\}$ are open or all $\alpha > 0$. Therefore $x \mapsto \mu\left(B(x,r)\right)$ and $x \mapsto \nu\left(B(x,r)\right)$ are measurable with respect to any Borel outer measure. Then

$$f_r(x) = \begin{cases} \dfrac{\nu(B(x,r))}{\mu(B(x,r))}, & \text{if } \mu\left(B(x,r)\right) > 0 \\ +\infty, & \text{if } \mu\left(B(x,r)\right) = 0. \end{cases}$$

is μ-measurable. Check it: Define $g_r : \{x : \mu\left(B(x,r)\right) > 0\} \to \mathbb{R}$ by

$$g_r(x) = \frac{\nu(B(x,r))}{\mu(B(x,r))}.$$

Define

$$X := \{x : \mu\left(B(x,r)\right) = 0\}.$$

Then g_r is μ-measurable on X^C.

First, show that $\mu(X) = 0$: If $x \in X$, we can find $\tilde{x} \in \mathbb{Q}^n$ with $|x - \tilde{x}| < \frac{r}{2}$. Then $x \in B(\tilde{x}, \frac{r}{2})$ and $B(\tilde{x}, \frac{r}{2}) \subset B(x, r)$, implying $\mu\left(B(\tilde{x}, \frac{r}{2})\right) = 0$. Now $X \subset \cup_{x \in X=1}^{\infty} B(\tilde{x}, \frac{r}{2})$ is a countable union of sets of measure zero and therefore

$$\mu(X) \leq \sum_{i=1}^{\infty} \mu\left(B\left(\tilde{x}_i, \frac{r}{2}\right)\right) = 0.$$

To show that f_r is μ-measurable on $\mathbb{R}^n$ we need to show that

$$Y := \{x \in \mathbb{R}^n : f_r(x) < t\}$$

is μ-measurable for all $t > 0$: Pick $T \in \mathbb{R}$. Then

$$\mu(T) = \mu(T \cap X) + \mu\left(T \cap X^C\right) = \mu\left(T \cap X^C\right).$$

But $Y \cap X^C$ is $\mu|_{X^C}$-measurable on X^C and therefore

$$\begin{aligned} \mu\left(T \cap X^C\right) &= \mu|_{X^C}(T) = \mu|_{X^C}(T \cap Y \cap X^C) + \mu|_{X^C}\left(T \cap (X^C - Y)\right) \\ &= \mu\left(T \cap Y \cap X^C\right) + \mu\left(T \cap X^C \cap Y^C\right) \\ &\quad + \mu(T \cap Y \cap X) + \mu\left(T \cap X \cap Y^C\right) \\ &= \mu(T \cap Y) + \mu\left(T \cap Y^C\right). \end{aligned}$$

$\therefore$ f_r is μ-measurable.

When the derivate exists $D_\mu \nu = \lim_{r \to 0} f_r$. So

$$D_\mu \nu = \lim_{r \to 0} f_r = \lim_{k \to \infty} f_{\frac{1}{k}} \quad \mu \text{ a.e.}$$

So $D_\mu \nu$ is μ-measurable.

For general μ, ν (not necessarily satisfying $\mu(\mathbb{R}^n), \nu(\mathbb{R}^n) < \infty$) we consider

$$\mu_k := \mu|_{B(0,k)} \ , \ \nu_k := \nu|_{B(0,k)}.$$

Then by the above argument, for all $k \in \mathbb{N}$, $D_{\mu_k} \nu_k$ exists and is finite μ_k a.e. However, for $x \in \text{int } B(0, k)$ such that $D_{\mu_k} \nu_k$ exists and is less than ∞, we have

$$D_\mu \nu(x) \text{ exists and } D_\mu \nu(x) = D_{\mu_k} \nu_k(x).$$

Therefore

$$\mu\left(\{x \in \text{int } B(0, k) : D_\mu \nu(x) \text{ does not exist or } D_\mu \nu(x) = \infty\}\right) = 0.$$

Letting $k \to \infty$ we obtain that $D_\mu \nu$ exists and is finite μ a.e. In addition,

$$\lim_{k \to \infty} D_{f_k} \nu_k = D_\mu \nu \quad \mu \text{ a.e.}$$

Thus $D_\mu \nu$ is μ-measurable. $\square_{\text{Claim 3}}$

$\square$

Definition 2.11 (Absolutely continuous measure)**.** An outer measure ν is called *absolutely continuous with respect to an outer measure* μ and written $\nu << \mu$ if $\mu(A) = 0$ implies $\nu(A) = 0$ for all $A \subset \mathbb{R}^n$.

Definition 2.12 (Mutually singular outer measures)**.** The outer measure μ and ν are called *mutually singular* and written $\nu \perp \mu$ if there exists a Borel set $B \subset \mathbb{R}^n$ such that

$$\mu\left(B^C\right) = \nu\left(B\right) = 0.$$

Theorem 2.13 (Fundamental Theorem of Calculus for Radon Outer Measures)**.** *Let μ, ν be Radon outer measures on $\mathbb{R}^n$ with $\mu << \nu$. Then*

$$\nu(A) = \int_A D_\mu \nu \ d\mu$$

for all μ-measurable sets $A \subset \mathbb{R}^n$.

Proof. Let A be μ-measurable. Then there exists a Borel set B with $A \subset B$ and $\mu\left(B - A\right) = 0$. So $\nu(B - A) = 0 \Rightarrow B - A$ is ν-measurable and $A = B - (B - A)$ is ν-measurable. Hence every μ-measurable set is also ν-measurable.

Define sets

$$\begin{aligned} Z &:= \{x \in \mathbb{R}^n : D_\mu\nu(x) = 0\} \\ I &:= \{x \in \mathbb{R}^n : D_\mu\nu(x) = +\infty\}. \end{aligned}$$

Since $D_\mu\nu$ is μ-measurable Z and I are μ-measurable. By Theorem 2.10, $\mu\left(I\right) = 0$ and so $\nu(I) = 0$. In addition by Theorem 2.9, for all $\alpha > 0$, we have

$$\nu(Z) \leq \alpha\mu\left(Z\right)$$

implying $\nu(Z) = 0$ and therefore

$$\nu(Z) = 0 = \int_Z D_\mu\nu \ \mathrm{d}\mu.$$

Also

$$\mu(I) = 0 = \int_I D_\mu\nu \ \mathrm{d}\mu.$$

Fix $1 < t < \infty$. Define for each $m \in \mathbb{Z}$

$$A_m = A \cap \left\{x \in \mathbb{R}^n : t^m \leq D_\mu\nu(x) < t^{m+1}\right\}.$$

The A_m are μ-measurable and ν-measurable. Moreover,

$$A - \cup_{m=-\infty}^{\infty} A_m \subset Z \cup I \cup \left\{x : \overline{D}_\mu\nu(x) \neq \underline{D}_\mu\nu(x)\right\}.$$

Since $\nu << \mu$, we obtain $\nu\left(A - \cup_{m=-\infty}^{\infty} A_m\right) = 0$ and therefore

$$\begin{aligned}
\nu(A) &\leq \sum_{m=-\infty}^{\infty} \nu(A_m) \leq \sum_{m=-\infty}^{\infty} t^{m+1}\mu\left(A_m\right) \\
&= t\sum_{m=-\infty}^{\infty} t^m \mu\left(A_m\right) \leq t\sum_{m=-\infty}^{\infty} \int_{A_m} D_\mu \nu \, \mathrm{d}\mu \\
&= t\int_A D_\mu \nu \, \mathrm{d}\mu.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nu(A) &= \sum_{m\in\mathbb{Z}} \nu(A_m) \geq \sum_{m\in\mathbb{Z}} t^m \mu(A_m) \\
&= \frac{1}{t}\sum_{m\in\mathbb{Z}} t^{m+1}\mu\left(A_m\right) \geq \frac{1}{t}\sum_{m\in\mathbb{Z}} \int_{A_m} D_\mu \nu \, \mathrm{d}\mu \\
&= \frac{1}{t}\int_A D_\mu \nu \, \mathrm{d}\mu.
\end{aligned}$$

Thus, for all $t \in (1, \infty)$, we obtain inequality

$$\frac{1}{t}\int_A D_\mu \nu \, \mathrm{d}\mu \leq \nu(A) \leq t\int_A D_\mu \nu \, \mathrm{d}\mu.$$

Letting $t \to 1^+$ we obtain the theorem. □

Theorem 2.14 (Lebesgue Decomposition Theorem)**.** *Let μ, ν be Radon outer measures on $\mathbb{R}^n$.*

(i) Then

$$\nu = \nu_{ac} + \nu_s,$$

where ν_{ac} and ν_s are Radon outer measures on $\mathbb{R}^n$ with $\nu_{ac} << \mu$ and $\nu_s \perp \mu$.

(ii) Furthermore, $D_\mu \nu = D_\mu \nu_{ac}$ and $D_\mu \nu_s = 0$ μ a.e. and consequently

$$\nu(A) = \nu_{ac}(A) + \nu_s(A) = \int_A D_\mu \nu_{ac} \; d\mu + \nu_s(A) = \int_A D_\mu \nu \; d\mu + \nu_s(A).$$

We call ν_{ac} the absolutely continuous part *and ν_s the* singular part*.*

Proof. Proof of part (i). Define

$$\mathcal{E} = \left\{A \subset \mathbb{R}^n : A \text{ Borel}, \mu\left(A^C\right) = 0\right\}.$$

Choose $B_k \in \mathcal{E}$ (not necessarily balls) such that for all $k \in \mathbb{N}$

$$\nu\left(B_k\right) \leq \inf_{A\in\mathcal{E}} \nu\left(A\right) + \frac{1}{k}.$$

Let

$$B := \cap_{k=1}^{\infty} B_k.$$

Since

$$\begin{aligned}\mu\left(B^C\right) &= \mu\left(\left(\cap_{k=1}^{\infty} B_k\right)^C\right) = \mu\left(\cup_{k=1}^{\infty} B_k^C\right) \\ &\leq \sum_{k=1}^{\infty} \mu\left(B_k^C\right) = 0,\end{aligned}$$

we have $B \in \mathcal{E}$.

For all $k \in \mathbb{N}$

$$\nu(B) \leq \nu(B_k) \leq \inf_{A \in \mathcal{E}} \nu(A) + \frac{1}{k}.$$

Letting $k \to \infty$, we obtain $\nu(B) \leq \inf_{A \in \mathcal{E}} \nu(A)$, since $B \in \mathcal{E}$, implying

$$\nu(B) = \inf_{A \in \mathcal{E}} \nu(A).$$

Define

$$\begin{aligned}\nu_{ac} &:= \nu|_B \\ \nu_s &:= \nu|_{B^C}.\end{aligned}$$

These are Radon outer measures by Theorem 1.13 and for all compact sets K,

$$\begin{aligned}\nu|_B(K) &= \nu(B \cap K) \leq \nu(K) < \infty, \\ \nu|_{B^C}(K) &= \nu\left(B^C \cap K\right) \leq \nu(K) < \infty.\end{aligned}$$

Suppose that $A \subset B$ such that A is Borel set and $\mu(A) = 0$ and $\nu(A) > 0$. Then

$$\begin{aligned}\mu\left((B - A)^C\right) &= \mu\left(\left(B \cap A^C\right)^C\right) = \mu\left(B^C \cup A\right) \\ &\leq \mu\left(B^C\right) + \mu(A) = 0,\end{aligned}$$

implying $B - A \in \mathcal{E}$ and

$$\nu(B - A) = \nu(B) - \nu(A) < \nu(B)$$

which is a contradiction. So $\mu(A) = 0 \Rightarrow \nu(A) = 0$ for all Borel sets $A \subset B$.

Since μ is Radon outer measure, for any set A there exists a Borel set $\tilde{A}$ such that $A \subset \tilde{A}$ and $\mu(A) = \mu\left(\tilde{A}\right)$. So if $\mu(A) = \mu\left(\tilde{A}\right) = 0$, then

$$\mu\left(\tilde{A} \cap B\right) = 0 \Rightarrow \mu\left(\tilde{A} \cap B\right) = 0 \Rightarrow \nu_{ac}(\tilde{A}) = 0 \Rightarrow \nu_{ac}(A) = 0$$

and therefore $\nu_{ac} << \mu$. Also $\mu\left(B^C\right) = 0$ and $\nu_s(B) = \nu|_{B^C}(B) = 0$, So $\mu \perp \nu_s$.

$\square_{\text{(i)}}$

Proof of part (ii). For any $\alpha > 0$, let

$$C := \{x \in B : D_\mu \nu_s(x) \geq \alpha\}.$$

By Theorem 2.9, $\alpha\mu(C) \leq \nu_s(C) = 0$. For all $\alpha > 0$ we then obtain

$$\begin{aligned} \mu\left(\{x : D_\mu \nu_s(x) \geq \alpha\}\right) &\leq \mu\left(B^C\right) + \mu\left(C\right) = 0 \\ \Rightarrow D_\mu \nu_s &= 0 \ \mu \text{ a.e..} \end{aligned}$$

Therefore

$$D_\mu \nu = D_\mu \nu_{ac} + D_\mu \nu_s = D_\mu \nu_{ac} \quad \mu \text{ a.e..}$$

$\square_{\text{(ii)}}$

$\square$

Theorem 2.15 (Lebesgue-Besicovitch Differentiation Theorem)**.** *Let μ be a Radon outer measure on $\mathbb{R}^n$ and $f \in L^1_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r \to 0} \frac{1}{\mu\left(B(x,r)\right)} \int_{B(x,r)} f \ d\mu = f(x)$$

for μ a.e. $x \in \mathbb{R}^n$.

Proof. For Borel set $B \subset \mathbb{R}^n$, let

$$\nu^\pm(B) := \int_B f^\pm \, \mathrm{d}\mu$$

and arbitrary $A \subset \mathbb{R}^n$

$$\nu^\pm(A) := \inf\left\{\nu^\pm(B) : A \subset B, B \text{ Borel}\right\}.$$

I.e. for each $k \in \mathbb{N}$, there exists B_k Borel such that $A \subset B_k$ and $\nu^\pm(B_k) \leq \nu^\pm(A) + \frac{1}{k}$. Let

$$\tilde{B}_k := \cap_{i=1}^k B_i \ , \ \tilde{B} := \cap_{i=1}^\infty B_i.$$

Now $\nu^\pm(\tilde{B}_k) \leq \nu^\pm(B_k) \leq \nu^\pm(A) + \frac{1}{k}$ and

$$\lim_{k \to \infty} \nu^\pm(\tilde{B}_k) = \nu^\pm(\tilde{B}) \leq \nu^\pm(A),$$

provided that $\nu^\pm(A) < \infty$ and for all $k \in \mathbb{N}$, $A \subset B_k$. Then

$$A \subset \tilde{B} \ \Rightarrow \ \nu^\pm(A) \leq \nu^\pm(\tilde{B}) \ \Rightarrow \ \tilde{B} \text{ is Borel } A \subset \tilde{B} \text{ and } \nu^\pm(A) = \nu^\pm(\tilde{B}).$$

If $\nu^\pm(A) = \infty$, then $A \subset \mathbb{R}^n$ and $\nu^\pm(A) = \nu^\pm(\mathbb{R}^n) = \infty$.
$\therefore$ $\nu^\pm$ is Borel regural.

Also for compact set $K \subset \mathbb{R}^n$ compact

$$\int_K f^\pm \, \mathrm{d}\mu < \infty,$$

since $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mu)$, implying that $\nu^{\pm}$ are Radon.

By Theorem 2.13

$$\nu^{\pm}(A) = \int_A f^{\pm} \, \mathrm{d}\mu = \int_A D_\mu \nu^{\pm} \, \mathrm{d}\mu$$

for all μ-measurable sets A. Thus

$$D_\mu \nu^{\pm} = f^{\pm} \quad \mu \text{ a.e.}$$

This implies

$$\begin{aligned}
\lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \int_{B(x,r)} f \, \mathrm{d}\mu &= \lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \left(\int_{B(x,r)} f^+ \, \mathrm{d}\mu - \int_{B(x,r)} f^- \, \mathrm{d}\mu \right) \\
&= \lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \left(\nu^+ \left(B(x,r)\right) - \nu^- \left(B(x,r)\right) \right) \\
&= D_\mu \nu^+(x) - D_\mu \nu^-(x) \\
&\overset{\mu \text{ a.e.}}{=} f^+(x) - f^-(x) \\
&= f(x)
\end{aligned}$$

for μ a.e. x. $\square$

Definition 2.16. We define $L^p_{\text{loc}}(\mathbb{R}^n, \mu)$ to be the set of all μ-measurable functions such that

$$\int_K |f|^p \, \mathrm{d}\mu < \infty$$

for any compact set $K \subset \mathbb{R}^n$. Here $1 \leq p < \infty$.

Corollary 2.17. *Let μ be a Radon outer measure on $\mathbb{R}^n$, $1 \leq p < \infty$ and $f \in L^p_{loc}(\mathbb{R}^n, \mu)$. Then*

$$\lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \int_{B(x,r)} |f(y) - f(x)|^p \, d\mu(y) = 0$$

for μ a.e. x.

Proof. Let $\{r_i\}_{i=1}^{\infty}$ be a countable dense subset of $\mathbb{R}$ (i.e. enumeration of the rationals). By Theorem 2.15, for each $i \in \mathbb{N}$ there exists $A_i \subset \mathbb{R}^n$ with $\mu(A_i) = 0$ and such that

$$\lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \int_{B(x,r)} |f - r_i|^p \, \mathrm{d}\mu = |f(x) - r_i|^p$$

for all $x \in A_i^C$. For $A := \cup_{i=1}^{\infty} A_i$ we have $\mu(A) = 0$ and for $x \in A^C$

$$\lim_{r\to 0} \frac{1}{\mu\left(B(x,r)\right)} \int_{B(x,r)} |f - r_i|^p \, \mathrm{d}\mu = |f(x) - r_i|^p$$

for all $i \in \mathbb{N}$.

Fix $x \in A^C$ and $\varepsilon > 0$. Choose r_i such that $|f(x) - r_i|^p < \frac{\varepsilon}{2^p}$. Then

$$\begin{aligned} & \limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - f(x)|^p \, \mathrm{d}\mu \\ \leq \; & 2^{p-1} \Big(\limsup_{r \to 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f - r_i|^p \, \mathrm{d}\mu \\ & + \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(x) - r_i|^p \, \mathrm{d}\mu \Big) \\ = \; & 2^{p-1} \left(|f(x) - r_i|^p + |f(x) - r_i|^p \right) < \varepsilon. \end{aligned}$$

□

Corollary 2.18. *If $f \in L^p_{loc}(\mathbb{R}^n, \mu)$ for some $1 \leq p < \infty$ then*

$$\lim_{B \searrow \{x\}} \frac{1}{\mathcal{L}^n(B)} \int_B |f(y) - f(x)| \, dy = 0$$

for $\mathcal{L}^n$ a.e. x, where the limit is taken over all closed balls (with $\operatorname{int} B \neq \emptyset$) containing x as $\operatorname{diam} B \to 0$.

Proof. Let $\{B_k\}_{k=1}^\infty$ be a sequence of closed balls with nonempty interior and

$$d_k := \operatorname{diam} B_k \to 0$$

as $k \to \infty$. Then $B_k \subset B(x, d_k)$ and $\mathcal{L}^n(B(x, d_k)) \leq 2^n \mathcal{L}^n(B_k)$, so

$$\frac{1}{\mathcal{L}^n(B_k)} \int_{B_k} |f(y) - f(x)|^p \, \mathrm{d}y \leq \frac{2^n}{\mathcal{L}^n(B(x, d_k))} \int_{B(x,d_k)} |f - f(x)|^p \, \mathrm{d}y \to 0$$

as $k \to \infty$ for $\mathcal{L}^n$ a.e. $x \in \mathbb{R}^n$. □

Corollary 2.19. *Let $E \subset \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then*

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 1$$

for $\mathcal{L}^n$ a.e. $x \in E$ and

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))} = 0$$

for $\mathcal{L}^n$ a.e. $x \in E^C$.

Proof. By Theorem 2.15, for $x \in E$ we have

$$1 = \chi_E(x) \overset{\mathcal{L}^n \text{ a.e.}}{=} \lim_{r \to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} \chi_E(y) \, \mathrm{d}y = \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap E)}{\mathcal{L}^n(B(x,r))}$$

for $\mathcal{L}^n$ a.e. $x \in E$. Similarly for $x \in E^C$ (still using χ_E). □

Definition 2.20 (Point of density 1 and 0)**.** Let $E \subset \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is a *point of density* 1 for E if

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))} = 1$$

and a point of density 0 for E if

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r)\cap E)}{\mathcal{L}^n(B(x,r))} = 0.$$

Definition 2.21 (Precise representative)**.** Let $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$, then

$$f^*(x) := \begin{cases} \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)\ \mathrm{d}y, & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is called the *precise representative of* f.

Remark 2.22. If $f, g \in L^1_{\mathrm{loc}}$ and $f = g \quad \mathcal{L}^n$ a.e. then

$$\frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)\ \mathrm{d}y = \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} g(y)\ \mathrm{d}y$$

and therefore

$$\lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} f(y)\ \mathrm{d}y = \lim_{r\to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} g(y)\ \mathrm{d}y.$$

The limit exists for presicely the same values of x and are equal. Therefore

$$f^*(x) = g^*(x)$$

for all $x \in \mathbb{R}^n$.

Definition 2.23. Let $f : \mathbb{R}^n \to \mathbb{R}^m$. We say $l \in \mathbb{R}^m$ is the *approximate limit of* f *as* $y \to x$, written

$$\operatorname{ap}\lim_{x\to y} f(y) = l.$$

if for each $\varepsilon > 0$

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(B(x,r)\cap \{y : |f(y) - l| \geq \varepsilon\}\right)}{\mathcal{L}\left(B(x,r)\right)} = 0.$$

In other words, if l is the approximate limit of f at x, for each $\varepsilon > 0$, x is a point of density 0 for the set $\{y : |f(y) - l| \geq \varepsilon\}$.

Theorem 2.24. *An approximate limit is unique.*

Proof. Assume for each $\varepsilon > 0$ that both

$$\lim_{r\to 0} \frac{\mathcal{L}^n\left(B(x,r)\cap \{y : |f(y) - l| \geq \varepsilon\}\right)}{\mathcal{L}\left(B(x,r)\right)} = 0$$

$$\lim_{r\to 0} \frac{\mathcal{L}^n \left(B(x,r) \cap \{y : |f(y) - l'| \geq \varepsilon\}\right)}{\mathcal{L}\left(B(x,r)\right)} = 0.$$

If $l \neq l'$ we take $\varepsilon := \frac{1}{3}|l - l'|$. Then for each $y \in B(x,r)$

$$3\varepsilon = |l - l'| \leq |f(y) - l| + |l - l'|$$

so either $|f(y) - l| \geq \varepsilon$ or $|f(y) - l'| > \varepsilon$. Therefore

$$\begin{aligned}
B(x,r) &\subset \{y : |f(y) - l| \geq \varepsilon\} \cup \{y : |f(y) - l'| \geq \varepsilon\} \\
\Rightarrow B(x,r) &\subset \left(B(x,r) \cap \{y : |f(y) - l| \geq \varepsilon\}\right) \\
&\quad \cup \left(B(x,r) \cap \{y : |f(y) - l'| \geq \varepsilon\}\right) \\
\Rightarrow \mathcal{L}^n(B(x,r)) &\leq \mathcal{L}^n \left(B(x,r) \cap \{y : |f(y) - l| \geq \varepsilon\}\right) \\
&\quad + \mathcal{L}^n \left(B(x,r) \cap \{y : |f(y) - l'| \geq \varepsilon\}\right)
\end{aligned}$$

implying

$$1 \leq \frac{\mathcal{L}^n \left(B(x,r) \cap \{y : |f(y) - l| \geq \varepsilon\}\right)}{\mathcal{L}^n(B(x,r))} + \frac{\mathcal{L}^n \left(B(x,r) \cap \{y : |f(y) - l'| \geq \varepsilon\}\right)}{\mathcal{L}^n(B(x,r))},$$

which is a contradiction. □

Definition 2.25 (Approximate lim sup and lim inf)**.** Let $f : \mathbb{R}^n \to R$. We can say that l is the *approximate* lim sup *of* f *as* $y \to x$, written

$$\operatorname{ap}\limsup_{y\to x} f(y) = l,$$

if l is the infimum of the real numbers t such that

$$\lim_{r\to 0} \frac{\mathcal{L}^n \left(B(x,r) \cap \{y : f(y) > t\}\right)}{\mathcal{L}^n(B(x,r))} = 0.$$

Similarly l is the *approximate* lim inf *of* f *as* $y \to x$, written

$$\operatorname{ap}\liminf_{y\to x} f(y) = l,$$

if l is the supremum of the real numbers t such that

$$\lim_{r\to 0} \frac{\mathcal{L}^n \left(B(x,r) \cap \{y : f(y) < t\}\right)}{\mathcal{L}^n(B(x,r))} = 0.$$

Definition 2.26 (Approximately continuous)**.** $f : \mathbb{R}^n \to \mathbb{R}^n$ is *approximately continuous* at $x \in \mathbb{R}^n$ if

$$\operatorname{ap}\lim_{y\to x} f(y) = f(x).$$

Theorem 2.27. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then f is approximately continuous $\mathcal{L}^n$ a.e.*

Proof. **Claim 1:** There exist disjoint, compact sets $\{K_i\}_{i=1}^{\infty} \subset \mathbb{R}^n$ such that

$$\mathcal{L}^n\left((\cup_{i=1}^{\infty} K_i)^l\right) = 0$$

and for each $i \in \mathbb{N}$, $f|_{K_i}$ is continuous.

Proof of Claim 1. For each m let

$$B_m := B(0, m).$$

By Lusin's Theorem, there exist a compact set $K_1 \subset B_m$ such that $\mathcal{L}^n(B_1 - K_1) \leq 1$ an $f|_{K_1}$ is continuous. Assume $K_1, \ldots, K_m$ have been constructed. There exists a compact set $K_{m+1} \subset B_{m+1} - \cup_{i=1}^{m} K_i$ such that $\mathcal{L}^n(B_{m+1} - \cup_{i=1}^{m+1} K_i) \leq \frac{1}{m+1}$ and $f|_{K_{m+1}}$ is continuous. Claim 1 proved. $\square_{\text{Claim 1}}$

We know from Corollary 2.19 that for $\mathcal{L}^n$ a.e. $x \in K_i$

$$\lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap K_i)}{\mathcal{L}^n(B(x,r))} = 0. \tag{2.1}$$

Define

$$A := \{x : x \in K_i \text{ for some } i \text{ and (2.1) holds}\}.$$

Then $\mathcal{L}^n(A^C) = 0$. Let $x \in A$ and fix $\varepsilon > 0$. Then there exists $i \in \mathbb{N}$ such that $x \in K_i$ and (2.1) holds. There also exists s such that $y \in K_i$ and $|x - y| < s$ imply $|f(x) - f(y)| < \varepsilon$ by continuity of $f|_{K_i}$. Then if $0 < r < s$,

$$B(x,r) \cap \{y : |f(y) - f(x)| \geq \varepsilon\} \subset B(x,r) \cap K_i^C$$

and therefore

$$\lim_{r \to 0} \frac{\mathcal{L}^n\left(B(x,r) \cap \{y : |f(y) - f(x)| \geq \varepsilon\}\right)}{\mathcal{L}^n(B(x,r))} \leq \lim_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap K_i^C)}{\mathcal{L}^n(B(x,r))}.$$

$\square$

3 Some connections to functional analysis and the Riesz Representation Theorem.

Definition 3.1. Let f be a real or extended real valued function on a topological space.

If $\{x : f(x) > \alpha\}$ is open for every α, f is called *lower semicontinuous.*

If $\{x : f(x) < \alpha\}$ is open for every α, f is called *upper semicontinuous.*

Remark 3.2. The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infimum of upper semicontinuous functions is upper semicontinuous.

Proof. If x is such that $\sup_\beta f_\beta(x) > \alpha$ then there exists β_x such that $f_{\beta_x}(x) > \alpha$ and

$$\left\{x : \sup_\beta f_\beta(x) > \alpha\right\}$$

is open. Similarly $\{x : \inf_\beta f_\beta(x) > \alpha\} = \cup_\beta \{x : f_\beta(x) > \alpha\}$ is open. □

Remark 3.3. If a function is both lower and upper semicontinuous then it is continuous.

Theorem 3.4 (Urysohn's Lemma)**.** *Suppose $V \subset \mathbb{R}^n$ is open and $K \subset V$ is compact. Then there exists $f \in C_c(\mathbb{R}^n)$ (a continuous function with a compact support) such that $0 \le f \le 1$, $\operatorname{spt} f \subset V$ and $f(x) = 1$ for all $x \in K$.*

Proof. **Claim 1:** Suppose $K \in \mathbb{R}^n$ is compact and $p \in K^C$. Then there exist open sets U and W such that $p \in U$, $K \subset W$ and $U \cap W = \emptyset$.

Proof of Claim 1. Since K is compact, $\operatorname{dist}(K, p) > 0$. Let

$$\varepsilon := \operatorname{dist}(K, p).$$

and

$$U := B\left(p, \frac{\varepsilon}{3}\right).$$

Now $\left\{\operatorname{int} B(q, \frac{\varepsilon}{3})\right\}_{q \in K}$ is an open cover of K, therefore there exists a finite subcover $\left\{\operatorname{int} B(q_i, \frac{\varepsilon}{3})\right\}_{i=1}^N$. Let

$$W := \cup_{i=1}^N \operatorname{int} B\left(q_i, \frac{\varepsilon}{3}\right)$$

which is open and $U \cap W = \emptyset$. $\square_{\text{Claim 1}}$

Claim 2: If $\{K_\alpha\}$ is a collection of compact subsets of $\mathbb{R}^n$ and if $\cap_\alpha K_\alpha = \emptyset$, then there exists a finite subcollection $K_{\alpha_1}, \ldots, K_{\alpha_N}$ such that

$$\cap_{i=1}^N K_i = \emptyset.$$

Proof of Claim 2. Let

$$V_\alpha := K_\alpha^C.$$

Fix $K_1 \in \{K_\alpha\}$. Since no point of K_1 belongs to every K_α, every point of K_1 belongs to some $K_\alpha^C = V_\alpha$. Therefore $\{V_\alpha\}$ is an open cover of K_1. Thus $K_1 \subset \cup_{i1}^N V_{\alpha_i}$ for some finite subcollection $\{V_{\alpha_i}\}_{i=1}^N$. This implies that

$$\emptyset = K_1 \cap \left(\cup_{i=1}^N V_{\alpha_i}\right)^C = K_1 \cap \left(\cap_{i=1}^N K_{\alpha_i}\right).$$

$\square_{\text{Claim 2}}$

Claim 3: Suppose U is open, $U \subset \mathbb{R}^n$, $K \subset U$, K is compact. Then there is an open set V with compact closure $\overline{V}$ such that $K \subset V \subset \overline{V} \subset U$.

Proof of Claim 3. There exists a finite collection of closed balls $B(x_i, 1)$, with $\{x_1, \ldots, x_N\} \subset K$ such that

$$K \subset \cup_{i=1}^N \operatorname{int} B(x_i, 1) =: G.$$

If $U = \mathbb{R}^n$, pick $V = G$. If $U \neq \mathbb{R}^n$, let

$$C := U^C \neq \emptyset.$$

By Claim 1, for each $p \in C$, there exists an open set W_p such that $K \subset W_p$ and $p \notin \overline{W}_p$. Hence

$$\left\{C \cap \overline{G} \cap \overline{W}_p\right\}_{p \in C}$$

is a collection of compact sets with empty intersection since $C \cap \left(\cap_{p\in C} \overline{W}_p\right) = \emptyset$. Therefore by Claim 2, there exists $p_1, \ldots, p_N \in C$ such that

$$C \cap \overline{G}\left(\cap_{i=1}^N \overline{W}_{p_i}\right) = \emptyset.$$

So we may choose $V = G \cap \left(\cap_{i=1}^N \overline{W}_{p_i}\right)$. Then V is open, $K \subset V$ and

$$\overline{V} \subset \overline{G} \cap \left(\cap_{i=1}^N \overline{W}_{p_i}\right) \subset C^C = U.$$

$\square_{\text{Claim 3}}$

Let $r_1 = 0$, $r_2 = 1$ and $r_3, r_4, \ldots$ be and enumeration of the rationals in $(0,1)$. By Claim 3, we can find open sets V_0, V_1 such that $\overline{V_0}$ and $\overline{V_1}$ are compact and

$$K \subset V_1 \subset \overline{V_1} \subset V_0 \subset \overline{V_0} \subset V.$$

Suppose $k \geq 2$ and $V_{r_1}, \ldots, V_{r_k}$ have been chosen in a way that $r_i < r_j$ implies $\overline{V_{r_j}} \subset V_{r_i}$. Then one of the rationals $r_1, \ldots, r_k$, say r_i will be the largest one which is smaller than r_{k+1} and another, say r_j, will be smallest one larger than r_{k+1}. Using Caim 3, we may find $V_{r_{k+1}}$ such that

$$\overline{V}_{r_j} \subset V_{r_{k+1}} \subset \overline{V}_{r_{k+1}} \subset V_{r_i}.$$

Continuing, we may obtain a collection $\{V_r\}_{r\in\mathbb{Q}\cap[0,1]}$ of open sets such that $K \subset V_1$, $\overline{V_0} \subset V$ and $\overline{V_r}$ is compact and $s > r$ implies $\overline{V_s} \subset V_r$.

Define

$$f_r(x) := \begin{cases} r, & \text{if } x \in V_r \\ 0, & \text{otherwise,} \end{cases}$$

$$g_s(x) := \begin{cases} 1, & \text{if } x \in \overline{V}_s \\ s, & \text{otherwise} \end{cases}$$

and

$$f := \sup_{r\in[0,1]\cap\mathbb{Q}} f_r, \quad g := \inf_{s\in[0,1]\cap\mathbb{Q}} g_s.$$

By Remark 3.2 f is lower semicontinuous and g is upper semicontinuous.

Clearly, $0 \leq f \leq 1$, $f(x) = 1$ if $x \in K$, $\operatorname{spt} f \subset \overline{V_0}$. It is enough now to show that $f = g$. Now

$$f_r(x) > g_s(x) \;\Rightarrow\; r > s,\; x \in V_r \text{ and } x \notin \overline{V}_s,$$

but $r > s$ implies $V_r \subset V_s$. Contradiction.
So $f_r \leq g_s$ for all r, s and therefore $f \leq g$.

Suppose $f(x) < g(x)$ for some x. Then there exist rationals r, s such that

$$f(x) < r < s < g(x).$$

Now

$$f(x) < r \;\Rightarrow\; x \notin V_r,$$

and

$$g(x) > s \;\Rightarrow\; x \in \overline{V}_s,$$

but $s > r$ implies $\overline{V}_s \subset V_r$. Contradiction.
$\therefore f = g$ and so f is continuous by Remark 3.3. □

Theorem 3.5. *Let μ be a Radon outer measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to [-\infty, \infty]$, $f \in L^1(\mathbb{R}^n, \mu)$. Then given $\varepsilon > 0$, we can find $g \in C_c(\mathbb{R}^n)$ such that*

$$\int |f - g| \; d\mu < \varepsilon.$$

Proof. By Theorem 1.28 (Approximation by Simple Functions) we can find a simple μ-measurable function with compact support h such that

$$\int |f - h| \; \mathrm{d}\mu \leq \varepsilon.$$

This is because we can split $f = f^+ - f^-$, $f^\pm \in L^1(\mathbb{R}^n, \mu)$, then approximate each $f^\pm$ as in Theorem 1.27 using a sequence of simple μ-measurable functions h_k such that $|h_k| \leq |f|$ and multiply h_k with $\chi_{B(0,k)}$. Then convergence of

$$\int \left|f - h_k\chi_{B(0,k)}\right| \; \mathrm{d}\mu$$

follows from the Dominated Convergence Theorem. So it is enough to apprximate the characteristic function of a bounded set by a continuous function with compact support.

Given χ_A, where A is μ-measurable, A is bounded and $\mu(A) < \infty$, there exists compact set K and open set U such that $K \subset A \subset U$ and $\mu(U - K) < \varepsilon$. By Theorem 3.4 (Urysohn's Lemma), there exists $f \in C_c(\mathbb{R}^n)$ with $0 \leq f \leq 1$, $\operatorname{spt} f \subset U$ and $f(x) = 1$ for all $x \in K$. Therefore

$$\int |\chi_A - f| \, \mathrm{d}\mu \leq \int_{U-K} |\chi_A - f| \, \mathrm{d}\mu \leq \mu(U - K) < \varepsilon.$$

□

Theorem 3.6 (Bounded Linear Extension)**.** *Let $T : D(T) \to Y$ be a bounded linear operator, where $D(T)$ lies in a normed space X and Y is a Banach space. Then T has an extension*

$$\tilde{T} : \overline{D(T)} \to Y,$$

where $\tilde{T}$ is a bounded linear operator.

Proof. Consider any $x \in \overline{D(T)}$. There is a sequence $(x_n) \subset D(T)$ such that $x_n \to x$. Then

$$||Tx_n - Tx_m|| = ||T(x_n - x_m)|| \leq C \, ||x_n - x_m|| \, .$$

So (Tx_n) is a Cauchy since (x_n) is Cauchy. Since Y is complete, (Tx_n) converges, say $Tx_n \to y \in Y$. Define

$$\tilde{T}x := y.$$

Now we need to show that the definition is independent of the sequene (x_n). Suppose $x_n \to x$ and $z_n \to X$. Let (v_n) be the sequence

$$(v_n)_{n=1}^{\infty} := (x_1, z_1, x_2, z_2, x_3, z_3, \ldots).$$

Then $v_n \to x$, (v_n) is Cauchy and therefore (Tv_n) converges (as above) and the subsequences (Tx_n) and (Tz_n) must converge to the same limit $y \in Y$.

Clearly $Tx = \tilde{T}x$ for all $x \in D(T)$ and since $||Tx_n|| \leq C \, ||x_n||$ for all $x \in D(T)$, letting $n \to \infty$ we obtain

$$\left|\left|\tilde{T}x\right|\right| \leq C \, ||x|| \, .$$

□

Definition 3.7 (Dual space, X^*)**.** Let X be a normed space. The set of all bounded linear functionals (linear map from a vector space to its field of scalars) on X, constitutes a normed space with norm defined by

$$||f|| = \sup_{\substack{x \in X \\ x \neq 0}} \frac{|f(x)|}{||x||} = \sup_{\substack{x \in X \\ ||x|| = 1}} |f(x)| \, ,$$

which is called the *dual space* of X, written X^*.

Definition 3.8 (Hilbert space)**.** A set $\mathcal{H}$ is a *Hilbert space*, if it satisfies the following:

(i) $\mathcal{H}$ is a vector space over $\mathbb{C}$ (or $\mathbb{R}$),

(ii) $\mathcal{H}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that

- $f \to \langle f, g \rangle$ is linear on $\mathcal{H}$ for every fixed $g \in \mathcal{H}$,
- $\langle f, g \rangle = \overline{\langle g, f \rangle}$, where $\overline{\langle g, f \rangle}$ is the complex conjugate of $\langle g, f \rangle$,
- $\langle f, f \rangle \geq 0$ for all $f \in \mathcal{H}$, we let

$$||f|| := \langle f, f \rangle^{\frac{1}{2}},$$

We also denote $\langle f, g \rangle = f \cdot g$.

(iii) $||f|| = 0 \Leftrightarrow f = 0$,

(iv) $\mathcal{H}$ is complete in the metric

$$d(f, g) := ||f - g|| .$$

Remark 3.9. The Cauchy-Schwartz inequality:

$$|\langle f, g \rangle| \leq ||f|| \, ||g||$$

The triangle inequality:

$$||f + g|| \leq ||f|| + ||g||$$

The parallelogram law:

$$||f + g||^2 + ||f - g||^2 = 2\left(||f||^2 + ||g||^2\right) \quad \text{for all } f, g \in \mathcal{H}$$

Follow from the definition, parts (i), (ii).

Theorem 3.10. *Suppose S is a closed subspace of $\mathcal{H}$ and $f \in \mathcal{H}$. Then*

(i) There exists a unique element $g_0 \in S$ which is closest to f in the sense that

$$||f - g_0|| = \inf_{g \in S} ||f - g|| .$$

(ii) The element $f - g_0$ is perpendicular *to S, that is,*

$$\langle f - g_0, g \rangle = 0 \quad \textit{for all } g \in S.$$

Proof. Proof of part (i). If $f \in S$, choose $f = g_0$. Otherwise, let

$$d := \inf_{g \in S} ||f - g|| .$$

Since S is closed and $f \notin S$, we have that $d > 0$. Let $(g_n)_{n=1} \subset S$ be such that $||f - g_n|| \to d$ as $n \to \infty$.

Claim: (g_n) is a Cauchy sequence.

Proof of Claim. By the parallelogram law, we have

$$||2f-(g_n+g_m)||^2+||g_n-g_m||^2=2\left(||f-g_n||^2+||f-g_m||^2\right).$$

Since S is a subspace, $\frac{1}{2}(g_n+g_m)\in S$ and therefore

$$||2f-(g_n+g_m)||=2\left|\left|f-\frac{1}{2}(g_n+g_m)\right|\right|\geq 2d.$$

So

$$\begin{aligned}||g_n-g_m||^2 &= 2\left(||f-g_n||^2+||f-g_m||^2\right)-||2f-(g_n+g_m)||^2\\ &\leq 2\left(||f-g_n||^2+||f-g_n||^2\right)-4d^2.\end{aligned}$$

Since $||f-g_n||\to d$ and $||f-g_m||\to d$ as $n,m\to\infty$, we have that (g_n) is Cauchy. $\square_{\text{Claim}}$

Since $\mathcal{H}$ is complete and S is closed $\liminf_{n\to\infty} g_n$ exists and is equal to $g_0\in S$. It also satisfies $||f-g_0||=d$. $\square_{\text{(i) Existence}}$

Proof of part (ii). Let $g\in S$. For each ε (negative or positive), consider $g_0-\varepsilon g\in S$. We have

$$||f-(g_0-\varepsilon g)||^2\geq ||f-g_0||^2.$$

And

$$||f-(g_0-\varepsilon g)||^2=||f-g_0||^2+\varepsilon^2||g||^2+2\varepsilon\operatorname{Re}\langle f-g_0,g\rangle,$$

implying

$$2\varepsilon\operatorname{Re}\langle f-g_0,g\rangle+\varepsilon^2||g||^2\geq 0.$$

If $\operatorname{Re}\langle f-g_0,g\rangle<0$ then taking ε small and positive leads to a contradiction. If $\operatorname{Re}\langle f-g_0,g\rangle>0$ then taking ε small and negative leads to contradiction.

$$\operatorname{Re}\langle f-g_0,g\rangle=0$$

Similarly, considering $g_0-i\varepsilon g$ we obtain

$$\operatorname{Im}\langle f-g_0,g\rangle=0,$$

therefore $\langle f-g_0,g\rangle=0$. $\square_{\text{(ii)}}$

Uniqueness in part (i). Suppose $\tilde{g}_0\neq g_0$ also satisfies

$$||f-\tilde{g}_0||=\inf_{g\in S}||f-g||.$$

Then

$$\langle f-g_0,g_0-\tilde{g}_0\rangle=0.$$

By the Pythagorean Theorem

$$||f-\tilde{g}_0||^2=||f-g_0||^2+||g_0-\tilde{g}_0||^2.$$

Since $||f-\tilde{g}_0||^2=||f-g_0||^2$, we have that $g_0=\tilde{g}_0$. $\square_{\text{(i) Uniqueness}}$

$\square$

Definition 3.11 (Orthogonal complement)**.** If S is a subspace of a Hilbert space $\mathcal{H}$, we define the *orthogonal complement* of S by

$$S^{\perp} := \{f \in \mathcal{H} : \langle f, g \rangle = 0 \text{ for all } g \in S\}.$$

$S^{\perp}$ is itself a subspace and $S \cap S^{\perp} = \{0\}$.

Theorem 3.12. *If S is a closed subspace of a Hilbert space $\mathcal{H}$, then*

$$\mathcal{H} = S \oplus S^{\perp}.$$

That is, every $f \in \mathcal{H}$ can be written uniquely and $f = g + h$, where $g \in S$ and $h \in S^{\perp}$.

Proof. For every $f \in \mathcal{H}$, choose g_0 as in the Theorem 3.10 and write $f = g_0 + (f - g_0)$. Then $g_0 \in S$ and $f - g_0 \in S^{\perp}$.

Suppose $f = g + h + \tilde{g} + \tilde{h}$ with $g, \tilde{g} \in S$ and $h, \tilde{h} \in S^{\perp}$. Then $g - \tilde{g} = \tilde{h} - h$. Since $g - \tilde{g} \in S$ and $h - \tilde{h} \in S^{\perp}$, we have $g - \tilde{g} = \tilde{h} - h = 0$. $\square$

Theorem 3.13. *Let L be a bounded linear functional on a Hilbert space $\mathcal{H}$. Then there exists a unique $g \in \mathcal{H}$ such that*

$$L(f) = \langle f, g \rangle \quad \textit{for all } f \in \mathcal{H}.$$

Moreover $||L|| = ||g||$.

Proof. Consider the subspace of $\mathcal{H}$ defined by

$$S := \{f \in \mathcal{H} : L(f) = 0\}$$

(called null-space of L, or kernel of L, $\ker L$). Since L is continuous, S is closed. If $S = \mathcal{H}$ then $L = 0$, pick $g = 0$. Otherwise, $S^{\perp} \neq \emptyset$, pick $h \in S^{\perp}$ with $||h|| = 1$. Let

$$g := \overline{L(h)}h,$$

where $\overline{L(h)}$ is the complex conjugate of $L(h)$. If we consider for any $f \in \mathcal{H}$, $L(f)h - L(h)f$, then

$$L\left(L(f)h - L(h)f\right) = L(f)L(h) - L(h)L(f) = 0,$$

since $L(f)$ is just a scalar. We then obtain

$$\begin{aligned} L(f)h - L(h)f \in S \ &\Rightarrow\ \langle L(f)h - L(h)f, h \rangle = 0 \\ &\Rightarrow\ L(f)\langle h, h \rangle - \left\langle f, \overline{L(h)}h \right\rangle = 0 \\ &\Rightarrow\ L(f) = \langle f, g \rangle. \end{aligned}$$

Now, $||L|| = \sup_{\substack{f \in \mathcal{H} \\ ||f||=1}} |L(f)| = \sup_{\substack{f \in \mathcal{H} \\ ||f||=1}} |\langle f, g \rangle| \leq ||g||$.

In addition, for $f = \frac{g}{||g||}$,

$$|L(f)| = |\langle f, g \rangle| = ||g||.$$

$\therefore ||L|| = ||g||$. $\square$

Definition 3.14 (Measurable function)**.** Given a measure space $(X, \mathcal{M}, \mu)$ we define the *measurable functions* $f : X \to [-\infty, \infty]$ with respect to $\mathcal{M}$ to be those f that satisfy

$$f^{-1}\left([-\infty, a)\right) = \{x \in X : f(x) < a\} \in \mathcal{M}$$

for all $a \in \mathbb{R}$.

Remark 3.15.

- Basic properties of measurable functions still hold (e.g. closure under algebraic manipulations, inf, sup, lim inf, lim sup).
- Notion of "almost everywhere" is with respect to μ.
- A simple function on X w.r.t. $\mathcal{M}$ takes the form

$$\sum_{k=1}^{N} a_k \chi_{E_k}$$

  where $E_k \in \mathcal{M}$.
- We can approximate measurable functions w.r.t. $\mathcal{M}$ by simple functions w.r.t. $\mathcal{M}$ in the same way as in Theorem 1.28.
- We define the *Lebesgue integral* in the same way, first for simple functions then for measurable functions $f : X \to [0, \infty]$ then for certain measurable $f : X \to [-\infty, \infty]$.
- Fatou's Lemma, monotone convergence and dominated convergence theorems hold.
- We define the spaces $L^p(X, \mu)$, $1 \le p < \infty$ to be the equivalence classes (modulo functions that vanish almost everywhere) of functions measurable w.r.t. $\mathcal{M}$ that satisfy

$$||f||_p := ||f||_{L^p(X,\mu)} := \left(\int_X |f(x)|^p \; \mathrm{d}\mu(x)\right)^{\frac{1}{p}} < \infty.$$

- The spaces $L^p(X, \mu)$, $1 \le p < \infty$ are complete normed spaces (Banach spaces) under the norm $||\cdot||_{L^p(X,\mu)}$.
- The space $L^2(X, \mu)$ is Hilbert space under the inner product

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)} \; \mathrm{d}\mu(x),$$

  where $\overline{g}$ is the complex conjugate of g.

Definition 3.16 (Signed measure)**.** Given a σ-algebra $\mathcal{M}$ on X, we define a *signed measure* ν on $\mathcal{M}$ to be a mapping $\nu : \mathcal{M} \to (-\infty, \infty]$ satisfying

(i) $\nu(\emptyset) = 0$

(ii) $\nu\left(\cup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \nu\left(E_i\right)$ whenever $\{E_i\}_{i=1}^{\infty}$ are disjoint sets in $\mathcal{M}$.

Definition 3.17 (Total variation)**.** Given a signed measure ν on a σ-algebra $\mathcal{M}$. We define *total variation* $|\nu|$ of ν to to be a function on $\mathcal{M}$ given by

$$|\nu|(E) = \sup\left\{\sum_{j=1}^{\infty} |\nu(E_j)| : E_j \in \mathcal{M},\ E = \bigcup_{j=1}^{\infty} E_j,\ E_j \text{ are disjoint}\right\}.$$

Theorem 3.18. *The total variation $|\nu|$ of a signed measure ν is a (positive) measure that satisfies*

$$\nu(A) \leq |\nu|(A).$$

for all $A \in \mathcal{M}$. We denote $\nu \leq |\nu|$.

Proof. Suppose $\{E_j\}_{j=1}^{\infty}$ are disjoint sets in $\mathcal{M}$ and $E = \cup_{j=1}^{\infty} E_j$.

Let $\alpha_j \in \mathbb{R}$ such that $\alpha_j < |\nu|(E_j)$. By Definition 3.17 for E_j we have

$$E_j = \cup_{i=1}^{\infty} F_{i,j}$$

with $F_{i,j}$ disjoint, $F_{i,j} \in \mathcal{M}$ for all $i, j \in \mathbb{N}$ and

$$\alpha_j \leq \sum_{i=1}^{\infty} |\nu(F_{i,j})|.$$

We have

$$\sum_{j=1}^{\infty} \alpha_j \leq \sum_{j,i=1}^{\infty} |\nu(F_{i,j})| \leq |\nu|(E)$$

since $E = \cup_{i,j=1}^{\infty} F_{i,j}$. Taking the supremum over all the numbers α_j, we have

$$\sum_{j=1}^{\infty} |\nu|(E_j) \leq |\nu|(E).$$

Now let $\{F_k\}_{k=1}^{\infty}$ be such that F_k are disjoint, $F_k \in \mathcal{M}$ for all $k \in \mathbb{N}$ and $E = \cup_{k=1}^{\infty} F_k$. For each k, $\{F_k \cap E_j\}_{j=1}^{\infty}$ are disjoint, $F_k \cap E_j \in \mathcal{M}$ and $F_k = \cup_{j=1}^{\infty} F_k \cap E_j$. So

$$\begin{aligned} \sum_{k=1}^{\infty} |\nu(F_k)| &= \sum_{k=1}^{\infty} \left| \sum_{j=1}^{\infty} \nu(F_k \cap E_j) \right| \\ &\leq \sum_{j,k=1}^{\infty} |\nu(F_k \cap E_j)| \\ &\leq \sum_{j=1}^{\infty} |\nu|(E_j). \end{aligned}$$

Taking supremum over all families $\{F_k\}_{k=1}^{\infty}$ that satisfy above conditions, we have

$$\begin{aligned} |\nu|(E) &\leq \sum_{j=1}^{\infty} |\nu|(E_j) \\ \Rightarrow |\nu|(E) &= \sum_{j=1}^{\infty} |\nu|(E_j). \end{aligned}$$

Clearly only $\emptyset \subset \emptyset \Rightarrow |\nu|(\emptyset) = \nu(\emptyset) = 0$. Since $E \subset E$, $\nu(E) \leq |\nu(E)| \leq |\nu|(E)$. □

Definition 3.19 (Positive and negative variation)**.** We define the *positive* and *negative variation* of ν by

$$\nu^+ = \frac{1}{2}(|\nu| + \nu) \quad \text{and} \quad \nu^- = \frac{1}{2}(|\nu| - \nu)$$

respectively.

Remark 3.20. ν^+, ν^- are measures and satisfy

$$\nu = \nu^+ - \nu^- \quad \text{and} \quad |\nu| = \nu^+ + \nu^-.$$

In the above, if $\nu(E) = +\infty$ for a set E, then $|\nu|(E) = +\infty$ and $\nu^-(E)$ is defined to be zero.

Definition 3.21 (σ-finite measure)**.** We say that the signed measure ν is *σ-finite* if the measure $|\nu|$ is σ-finite i.e. the underlying space X is a union of countably many sets in the underlying σ-algebra $\mathcal{M}$ of finite $|\nu|$-measure. If ν is σ-finite then so are ν^+ and ν^-.

Definition 3.22 (Mutually singular and absolutely continuous measures)**.** Two signed measures ν and μ on $(X, \mathcal{M})$ are *mutually singular* and we write $\nu \perp \mu$, if there are disjoint subsets A and B in $\mathcal{M}$ such that

$$\nu(E) = \nu(A \cap E) \quad \text{and} \quad \mu(E) = \mu(B \cap E)$$

for all $E \in \mathcal{M}$. If ν is a signed measure and μ a (positive) measure on $\mathcal{M}$, we say that ν is *absolutely continuous* with respect to μ if

$$E \in \mathcal{M} \text{ and } \mu(E) = 0 \Rightarrow \nu(E) = 0.$$

In this case we write $\nu << \mu$.

Theorem 3.23 (Radon-Nikodym Theorem)**.** *Suppose μ is a σ-finite (positive) measure on the measure space $(X, \mathcal{M}, \mu)$ and ν is a σ-finite signed measure on $\mathcal{M}$. Then there exist unique signed measures ν_{ac} and ν_s such that $\nu_{ac} << \mu$, $\nu_s \perp \mu$ and $\nu = \nu_{ac} + \nu_s$. In addition the signed measure ν_{ac} takes the form*

$$\nu_{ac}(E) = \int_E f(x)\, d\mu(x)$$

for some f measurable with respect to $\mathcal{M}$ and for all $E \in \mathcal{M}$.

Proof. Assume first that both ν and μ are positive and finite. Let $\rho = \nu + \mu$ and consider the mapping l on $L^2(X, \rho)$ defined by

$$l(\Psi) := \int_X \Psi(x) \, \mathrm{d}\nu(x)$$

for all $\Psi \in L^2(X, \rho)$. Then $l \in (L^2(X, \rho))^*$ because

$$\begin{aligned} |l(\Psi)| &\leq \int_X |\Psi(x)| \, \mathrm{d}\nu(x) \leq \int_X |\Psi(x)| \cdot 1 \, \mathrm{d}\rho(x) \\ &\leq (\rho(x))^{\frac{1}{2}} \left(\int_X |\Psi(x)|^2 \, \mathrm{d}\rho(X) \right)^{\frac{1}{2}} \end{aligned}$$

by the Cauchy-Schwarz inequality. Since $L^2(X, \rho)$ is a Hilbert space, Theorem 3.13 gives us the existence of some $g \in L^2(X, \rho)$ such that

$$\begin{aligned} l(\Psi) &= \int_X \Psi(x) \, \mathrm{d}\nu(x) = \int_X \Psi(x) g(x) \, \mathrm{d}\rho(x) && (3.1) \\ &= \int_X \Psi(x) g(x) \, \mathrm{d}\nu(x) + \int_X \Psi(x) g(x) \, \mathrm{d}\mu(x) && (3.2) \end{aligned}$$

for all $\Psi \in L^2(X, \rho)$. Hence, we may assume $0 \leq g(x) \leq 1$ for all $x \in X$.

We have

$$\int \Psi \cdot (1 - g) \, \mathrm{d}\nu = \int \Psi g \, \mathrm{d}\mu.$$

Consider the sets

$$\begin{aligned} A &:= \{x \in X : 0 \leq g(x) < 1\} \\ B &:= \{x \in X : g(x) = 1\}. \end{aligned}$$

Let

$$\nu_{ac}(E) := \nu(A \cap E) \quad , \quad \nu_s(E) := \nu(B \cap E).$$

For $\Psi = \chi_B$, $\int_B 1 - g \, \mathrm{d}\nu = 0 = \int_B g \, \mathrm{d}\mu = \mu(B)$. Therefore $\mu(E) = \mu\left(B^C \cap E\right)$ for all $E \in \mathcal{M}$, so

$$\nu_s \perp \mu.$$

Now let

$$\Psi := \chi_E(1 + g + \ldots + g^n).$$

Then $\int_E (1-g)(1 + g + \ldots + g^n) \, \mathrm{d}\nu = \int_E 1 - g^{n+1} \, \mathrm{d}\nu = \int_E g(1 + g + \ldots + g^n) \, \mathrm{d}\mu$. Since $(1 - g^{n+1})(x) = 0$ for all $x \in B$ and $\lim_{n\to\infty}(1 - g^{n+1})(x) = 1$ for all $x \in A$ by The Dominated Convergence Theorem, we have that

$$\lim_{n\to\infty} \int_E 1 - g^{n+1} \, \mathrm{d}\nu = \int_E \chi_A \, \mathrm{d}\nu = \nu(A \cap E) = \nu_{ac}(E).$$

Also for all $x \in A$,

$$\lim_{n\to\infty} g(1 + g + \ldots + g^n)(x) = \frac{g(x)}{1 - g(x)} =: f(x).$$

So $\lim_{n\to\infty} \int_E g(1+g+\ldots+g^n)\ \mathrm{d}\mu = \int_E f\ \mathrm{d}\mu$ and therefore $\nu_{ac}(E) = \int_E f\ \mathrm{d}\mu$ for all $E \in \mathcal{M}$. Since $f \geq 0$ and $\int_X f\ \mathrm{d}\mu = \nu_{ac}(X) \leq \nu(X) < \infty$, in this case $f \in L^1(X,\mu)$. Since $\mu(E) = 0$ we have $\int_E f\ \mathrm{d}\mu = 0 = \nu_{ac}(E)$ so

$$\nu_{ac} << \mu.$$

$\therefore$ ν_{ac} and ν_s are found for positive and finite ν and μ.

Consider the case where μ and ν are positive and σ-finite. We can find disjoint sets $E_j \in \mathcal{M}$ such that $X = \cup_{j=1}^{\infty} E_j$ and $\mu(E_j) < \infty$, $\nu(E_j) < \infty$, for all $j \in \mathbb{N}$. We then define, for any $E \in \mathcal{M}$

$$\begin{aligned} \mu_j(E) &:= \mu(E \cap E_j), \\ \nu_j(E) &:= \nu(E \cap E_j), \end{aligned}$$

which are positive and finite and, for each $j \in \mathbb{N}$, we have $\nu_j = \nu_{j_{\mathrm{ac}}} + \nu_{j_{\mathrm{s}}}$ where $\nu_{j_{\mathrm{s}}} \perp \mu_j$ and for all $E \in \mathcal{M}$

$$\nu_{j_{\mathrm{ac}}}(E) = \int_E f_j\ \mathrm{d}\mu_j.$$

We then set

$$f := \sum_{j=1}^{\infty} f_j \quad , \quad \nu_s := \sum_{j=1}^{\infty} \nu_{j_{\mathrm{s}}} \quad , \quad \nu_{ac} := \sum_{j=1}^{\infty} \nu_{j_{\mathrm{ac}}}.$$

Finally, if ν is signed, we apply the previous argument to ν^+ and ν^-.

To prove uniqueness, suppose in addition $\nu = \nu'_{ac} + \nu'_s$, where $\nu'_{ac} << \mu$, $\nu'_s \perp \mu$. then $\nu_{ac} - \nu'_{ac} = \nu'_s - \nu_s$ and both side have to be both absolutely continuous and singular with respect to μ implying

$$\nu_{ac} - \nu'_{ac} = \nu'_s - \nu_s = 0.$$

$\square$

Definition 3.24 ($L^\infty(X,\mu)$)**.** Given a measure space $(X,\mathcal{M},\mu)$ we define $L^\infty(X,\mu)$ to be the space of all μ-measurable functions on X such that

$$||f||_\infty := \inf\{\alpha : \mu(\{x \in X : |f(x)| > \alpha\}) = 0\} < \infty.$$

Remark 3.25.

$$\mu(\{x : |f(x)| > ||f||_\infty\}) = 0.$$

Proof. Let

$$F_k := \left\{x : |f(x)| > ||f||_\infty + \frac{1}{k}\right\}.$$

We have $\mu(F_k) = 0$ for all $k \in \mathbb{N}$. Therefore

$$\mu(\{x : |f(x)| > ||f||_\infty\}) = \mu(\cup_{k=1}^{\infty} F_k) = 0.$$

$\square$

Theorem 3.26. *Given finite outer measue μ on $\mathbb{R}^n$, there is a bijective isometry (bijective mapping that preserves the norm) between the spaces $L^\infty(\mathbb{R}^n,\mu)$ and $(L^1(\mathbb{R}^n,\mu))^*$. In particular, every $F \in (L^1(\mathbb{R}^n,\mu))^*$ has the form*

$$F(f) = \int_{\mathbb{R}^n} fg \ d\mu$$

for some $g \in L^\infty(\mathbb{R}^n,\mu)$.

Proof. Define, for every $g \in L^\infty(\mathbb{R}^n,\mu)$

$$\lambda_g(f) := \int fg \ \mathrm{d}\mu$$

for all $f \in L^1(\mathbb{R}^n,\mu)$. Then

$$\begin{aligned} ||\lambda_g|| &= \sup_{\substack{f\in L^1(\mathbb{R}^n,\mu)\\ ||f||_1=1}} |\lambda_g(f)| \le \sup_{\substack{f\in L^1(\mathbb{R}^n,\mu)\\ ||f||_1=1}} \int |fg| \ \mathrm{d}\mu \\ &\le ||g||_\infty \sup_{\substack{f\in L^1(\mathbb{R}^n,\mu)\\ ||f||_1=1}} \int |f| \ \mathrm{d}\mu = ||g||_\infty . \end{aligned}$$

Linearity is obvious from the definition, therefore $\lambda_g \in (L^1(\mathbb{R}^n,\mu))^*$.

To show that λ is an isometry. That is $||\lambda_g|| = ||g||_\infty$. For any $\varepsilon > 0$, the set

$$E := \{x : |g(x)| \ge ||g||_\infty - \varepsilon\}$$

has $\mu(E) > 0$. So we may construct some $f \ge 0$ such that $||f||_1 = 1$ and $f(x) = 0$ for all $x \in E^C$. In particular, take

$$f(x) := \begin{cases} (\mu(E))^{-1} & \text{for } x \in E \\ 0 & \text{for } x \in E^C. \end{cases}$$

Then

$$\begin{aligned} \lambda_g(f \operatorname{sign} g) &= \int_E f \underbrace{g \operatorname{sign} g}_{=|g|} \ \mathrm{d}\mu \ge (||g||_\infty - \varepsilon)\, ||f||_1 \\ &= ||g||_\infty - \varepsilon, \end{aligned}$$

where $\operatorname{sign} g = \frac{g}{|g|}$. Since $||f \operatorname{sign} g||_1 = 1$, we have $||\lambda_g|| = ||g||_\infty$.

To prove surjectivity. Given any $F \in (L^1(\mathbb{R}^n,\mu))^*$ define signed measure ν on the σ-algebra of all μ-measurable sets, $\mathcal{M}$, as follows:
For any $E \in \mathcal{M}$, $\nu(E) = F(\chi_E)$.
Clearly $\nu(\emptyset) = F(\chi_\emptyset) = 0$. Additivy follows from the linearity of F.

$$|\nu(E)| = |F(\chi_E)| \le C\, ||\chi_E||_{L^1} = C\mu(E).$$

So $\mu(E) = 0$. Therefore $\nu(E) = 0$, so $\nu << \mu$.

Now by Theorem 3.23 there exists h measurable w.r.t. $\mathcal{M}$ such that

$$F(\chi_E) = \nu(E) = \int_E h \, \mathrm{d}\mu$$

for all $E \in \mathcal{M}$. This implies that for any simple function f

$$F(f) = \int_{\mathbb{R}^n} fh \, \mathrm{d}\mu.$$

Let

$$A := \{x \in \mathbb{R}^n : |h(x)| > ||F||\}$$

be such that $\mu(A) > 0$. This implies $||h||_\infty > ||F||$. Consider simple μ-measurable function $\frac{\operatorname{sign} h}{\mu(A)}\chi_A$. Then

$$\left|\left|\frac{\operatorname{sign} h}{\mu(A)}\chi_A\right|\right|_1 = \int_{\mathbb{R}^n} \left|\frac{\operatorname{sign} h}{\mu(A)}\chi_A\right| \, \mathrm{d}\mu = 1$$

and

$$\left|F\left(\frac{\operatorname{sign} h}{\mu(A)}\chi_A\right)\right| = \frac{1}{\mu(A)} \int_A |h| \, \mathrm{d}\mu > ||F|| \, .$$

which is a contradiction.

$\therefore h \in L^\infty(\mathbb{R}^n, \mu)$ and $||h||_\infty \leq ||F||$.

Finally by the Dominated Convergence Theorem and the continuity of F on $L^1(\mathbb{R}^n, \mu)$ we have that

$$F(f) = \int_{\mathbb{R}^n} fh \, \mathrm{d}\mu$$

for all $f \in L^1(\mathbb{R}^n, \mu)$. Therefore $\lambda_h = F$. □

Corollary 3.27. *Let μ be a σ-finite outer measure on $\mathbb{R}^n$. There is a bijective isometry between the spaces $L^\infty(\mathbb{R}^n, \mu)$ and $(L^1(\mathbb{R}^n, \mu))^*$. In particular, for any $F \in (L^1(\mathbb{R}^n, \mu))^*$, we have*

$$F(f) = \int_{\mathbb{R}^n} fg \; d\mu$$

for some $g \in L^\infty(\mathbb{R}^n, \mu)$.

Proof. (Modification of the proof of Theorem 3.26.) Define for $g \in L^\infty(\mathbb{R}^n, \mu)$

$$\lambda_g(f) := \int fg \, \mathrm{d}\mu$$

for all $f \in L^1(\mathbb{R}^n, \mu)$. Then $\lambda_g \in (L^1(\mathbb{R}^n, \mu))^*$, $\lambda : L^\infty(\mathbb{R}^n, \mu) \to (L^1(\mathbb{R}^n, \mu))^*$.

To show that λ is an isometry, use σ-finiteness to split $\mathbb{R}^n$ to disjoint, μ-measurable sets X_j with $\mu(X_j) < \infty$ and $\mathbb{R}^n = \cup_{j=1}^\infty X_j$. Then substitute set E in Theorem 3.26 by

$$E_j := \{x \in X_j : |g(x)| \geq ||g||_\infty - \varepsilon\}$$

such that $\mu(E_j) > 0$ for all $j \in \mathbb{N}$.

To show surjectivity. Given $F \in (L^1(\mathbb{R}^n, \mu))^*$ define

$$\begin{aligned} \mu_j &:= \mu|_{X_j}, \\ f_j &:= f\chi_{X_j}. \end{aligned}$$

Consider $F_j : L^1(\mathbb{R}^n, \mu) \to \mathbb{R}$, defined by

$$F_j(f) := F(f_j).$$

Note that $L^1(\mathbb{R}^n, \mu) \subset L^1(\mathbb{R}^n, \mu_j)$ for all $j \in \mathbb{N}$ and

$$F(f) = F\left(\sum_{j=1}^{\infty} f_j\right) = \sum_{j=1}^{\infty} F(f_j) = \sum_{j=1}^{\infty} F_j(f)$$

for all $f \in L^1(\mathbb{R}^n, \mu)$. Also note that, for all $j \in \mathbb{N}$,

$$\begin{aligned} ||F_j|| &= \sup_{\substack{f \in L^1(\mathbb{R}^n, \mu_j) \\ ||f||_{L^1(\mathbb{R}^n, \mu_j)} = 1}} |F_j(f)| = \sup_{\substack{f \in L^1(\mathbb{R}^n, \mu_j) \\ ||f||_{L^1(\mathbb{R}^n, \mu_j)} = 1}} |F(f_j)| \\ &= \sup_{\substack{f \in L^1(\mathbb{R}^n, \mu_j) \\ \left|\left|f\chi_{X_j}\right|\right|_{L^1(\mathbb{R}^n, \mu_j)} = 1}} \left|F(f\chi_{X_j})\right| = \sup_{\substack{f \in L^1(\mathbb{R}^n, \mu_j) \\ \operatorname{spt} f \subset X_j \\ \left|\left|f\chi_{X_j}\right|\right|_{L^1(\mathbb{R}^n, \mu_j)} = 1}} |F(f)| \\ &\leq ||F||. \end{aligned}$$

By Theorem 3.26, there exists $h_j \in L^\infty(\mathbb{R}^n, \mu_j)$ such that

$$F_j(f) = \int_{\mathbb{R}^n} f h_j \, \mathrm{d}\mu_j = \int_{X_j} f h_j \, \mathrm{d}\mu$$

for all $f \in L^1(\mathbb{R}^n, \mu_j)$. We may assume that $\operatorname{spt} h_j \subset X_j$. Then for any $f \in L^1(\mathbb{R}^n, \mu)$ we have

$$F(f) = \sum_{j=1}^{\infty} F_j(f) = \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} f h_j \, \mathrm{d}\mu = \int_{\mathbb{R}^n} f h \, \mathrm{d}\mu$$

where $h = \sum_{j=1}^{\infty} h_j$. In addition

$$||h||_\infty \leq \sup_{1 \leq j < \infty} ||h_j||_\infty \leq \sup_{1 \leq j < \infty} ||F_j||_\infty \leq ||F||.$$

□

Theorem 3.28 (Riesz Representation Theorem)**.** *Let $L : C_c(\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\{L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m), |f| \leq 1, \operatorname{spt} f \subset K\} < \infty,$$

for each compact set $K \subset \mathbb{R}^n$. Then there exists a Radon outer measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that *(i)* $|\sigma(x)| = 1$ *for* μ *a.e.* x *and*

(ii) $L(f) = \int_{\mathbb{R}^n} \langle f, \sigma \rangle \; d\mu = \int_{\mathbb{R}^n} f \cdot \sigma \; d\mu$ *for all* $f \in C_c(\mathbb{R}^n; \mathbb{R}^m)$.

Note that here notations $|f|$ and $|\sigma(x)|$ are used for the Euclidean norm and $\cdot$ for the inner product.

Proof. We define the *variation outer measure* μ as follows. First for any open set $V \subset \mathbb{R}^n$ by

$$\mu(V) := \sup \{L(f) : f \in C_c(\mathbb{R}^n, \mathbb{R}^m), |f| \leq 1, \operatorname{spt} f \subset V\}$$

and for arbitrary $A \subset \mathbb{R}^n$,

$$\mu(A) := \inf \{\mu(V) : V \text{ is open s.t } A \subset V\}.$$

Claim 1: μ is an outer measure.

Proof of Claim 1. Let V, $\{V_i\}_{i=1}^{\infty}$ be open subset of $\mathbb{R}^n$ and $V \subset \cup_{i=1}^{\infty} V_i$. Choose $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ such that $|g| \leq 1$ and $\operatorname{spt} g \subset V$. Because $\operatorname{spt} g$ is compact, there exists $k \in \mathbb{N}$ such that $\operatorname{spt} g \subset \cup_{j=1}^{k} V_j$.

Let $\{J_j\}_{j=1}^{k}$ be smooth functions such that $\operatorname{spt} J_j \subset V_j$ for $1 \leq j \leq k$ and $\sum_{j=1}^{k} J_j = 1$ on $\operatorname{spt} g$. Then

$$|L(g)| = \left| \sum_{j=1}^{k} L(gJ_j) \right| \leq \sum_{j=1}^{k} |L(gJ_j)| \leq \sum_{j=1}^{\infty} \mu(V_j).$$

Then, taking supremum over g we have

$$\mu(V) \leq \sum_{j=1}^{\infty} \mu(V_j). \tag{3.3}$$

For arbitrary sets A, $\{A_j\}_{j=1}^{\infty}$ with $A \subset \cup_{j=1}^{\infty} A_j$, fix $\varepsilon > 0$. Then choose open sets V_j such that $A_j \subset V_j$ and $\mu(A_j) + \varepsilon 2^{-j} \geq \mu(V_j)$. Then

$$\mu(A) \leq \mu\left(\cup_{j=1}^{\infty} V_j\right) \leq \sum_{j=1}^{\infty} \mu(V_j) \leq \sum_{j=1}^{\infty} \mu(A_j) + \varepsilon \sum_{j=1}^{\infty} 2^{-j},$$

implying

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

$\square_{\text{Claim 1}}$

Claim 2: μ is Radon outer measure.

Proof of Claim 2. Let U_1 and U_2 be open sets with $\mathrm{dist}(U_1, U_2) > 0$. Then, given any $\varepsilon > 0$, there exists, $f_1, f_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $|f_{1,2}| \leq 1$ and $\mathrm{spt}\, f_{1,2} \subset U_{1,2}$ respectively and such that

$$\left.\begin{array}{rcl} L(f_1) & \geq & \mu(U_1) - \varepsilon \\ L(f_2) & \geq & \mu(U_2) - \varepsilon \end{array}\right\} \Rightarrow L(f_1 + f_2) \geq \mu(U_1) + \mu(U_2) - 2\varepsilon.$$

Then $f := f_1 + f_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$, $|f| \leq 1$ and $\mathrm{spt}\, f \subset U_1 \cup U_2$, since $\mathrm{dist}(U_1, U_2) > 0$. Also

$$\begin{aligned} \mu(U_1 \cup U_2) \geq L(f) & \geq \mu(U_1) + \mu(U_2) - 2\varepsilon \quad \forall \varepsilon. \\ \Rightarrow \mu(U_1 \cup U_2) & \geq \mu(U_1) + \mu(U_2) \qquad (3.4) \\ \Rightarrow \mu(U_1 \cup U_2) & = \mu(U_1) + \mu(U_2), \end{aligned}$$

where the last equality follows from (3.3) and (3.4).

For arbitrary $A_1, A_2 \subset \mathbb{R}^n$ with $\mathrm{dist}(A_1, A_2) > 0$. Given any $\varepsilon > 0$, there exists U_ε such that $A_1 \cup A_2 \subset U_\varepsilon$ and $\mu(A_1 \cup A_2) + \varepsilon \geq \mu(U_\varepsilon)$. Let

$$\delta := \frac{1}{3} \mathrm{dist}(A_1, A_2)$$

and define the δ-*enlargements*

$$A_1^\delta := \bigcup_{x \in A_1} \mathrm{int}\, B(x, \delta) \quad , \quad A_2^\delta := \bigcup_{x \in A_2} \mathrm{int}\, B(x, \delta).$$

Then A_1^δ, A_2^δ are open, $A_1 \subset A_1^\delta$, $A_2 \subset A_2^\delta$ and $\mathrm{dist}(A_1^\delta, A_2^\delta) > 0$. So for any $\varepsilon > 0$

$$\begin{aligned} \mu(A_1 \cup A_2) + \varepsilon & \geq \mu(U_\varepsilon) \geq \mu\left(U_\varepsilon \cap \left(A_1^\delta \cup A_2^\delta\right)\right) \\ & = \mu\left(\left(A_1^\delta \cap U_\varepsilon\right) \cup \left(A_2^\delta \cap U_\varepsilon\right)\right) \\ & = \mu\left(\left(A_1^\delta \cap U_\varepsilon\right)\right) + \mu\left(\left(A_2^\delta \cap U_\varepsilon\right)\right) \\ & \geq \mu(A_1) + \mu(A_2), \end{aligned}$$

since $A_1 \subset A_1^\delta \cap U_\varepsilon$, $A_2 \subset A_2^\delta \cap U_\varepsilon$. Therefore $\mu(A_1 \cup A_2) \geq \mu(A_1) + \mu(A_2)$. By Caratheodory's Criterion (Theorem 1.17) μ is Borel.
$\therefore$ μ is Borel.

Furthermore, since for arbitrary A,

$$\mu(A) = \inf\{\mu(U) : U \text{ open s.t. } A \subset U\}$$

there exist open sets $\{U_k\}_{k=1}^\infty$ such that $A \subset U_k$ and

$$\mu(U_k) \leq \mu(A) + \frac{1}{k}$$

for all $k \in \mathbb{N}$. Then $\mu(A) = \mu(\cap_{k=1}^\infty U_k)$ and $\cap_{k=1}^\infty U_k$ is Borel so μ is Borel regular.
$\therefore$ μ is Borel regular.

Let K be compact. Given an open cover of K by sets $\operatorname{int} B(x,1)$ where $x \in K$ we can find a finite subcover $\operatorname{int} B(x_i,1)$, where $1 \le i \le N$ and $x_i \in K$. Remember, that for each compact set K,

$$\mu(K) = \inf\{\mu(U) : U \text{ open s.t. } K \subset U\}.$$

So if $\mu(K) = \infty$ then $\mu\left(\cup_{i=1}^N \operatorname{int} B(x_i,1)\right) = \infty$. So for any $n \in \mathbb{N}$ there exists $f_n \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $|f_n| \le 1$ and $\operatorname{spt} f_n \subset \cup_{i=1}^N \operatorname{int} B(x_i,1) \subset \cup_{i=1}^N B(x_i,1)$, where $\cup_{i=1}^N B(x_1,1)$ is compact, and such that

$$L(f_n) \ge n.$$

This is a contradiction. Thus $\mu(K) < \infty$ for a compact set K.
$\therefore$ μ is Radon. $\square_{\text{Claim 2}}$

Now let

$$C_c^+(\mathbb{R}^n) := \{f \in C_c(\mathbb{R}^n) : f \ge 0\}$$

and let

$$\lambda(f) := \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \le f\}$$

If $f_1, f_2 \in C_c^+(\mathbb{R}^n)$ such that $f_1 \le f_2$, then all $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ satisfying $|g| \le f_1$ also satisfy $|g| \le f_2$. Therefore

$$\lambda(f_1) \le \lambda(f_2).$$

In addition for all $c \ge 0$ and $f \in C_c^+(\mathbb{R}^n)$

$$\lambda(cf) = c\lambda(f).$$

Claim 3: For all $f_1, f_2 \in C_c^+(\mathbb{R}^n)$,

$$\lambda(f_1 + f_2) = \lambda(f_1) + \lambda(f_2).$$

Proof of Claim 3. If $g_1, g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $|g_1| \le f_1$ and $|g_2| \le f_2$, then $|g_1 + g_2| \le f_1 + f_2$.

We may assume $L(g_1), L(g_2) \ge 0$. Therefore

$$|L(g_1)| + |L(g_2)| = L(g_1 + g_2) = |L(g_1 + g_2)| \le \lambda(f_1 + f_2).$$

Taking supremum over g_1 and $g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ we have

$$\lambda(f_1) + \lambda(f_2) \le \lambda(f_1 + f_2). \tag{3.5}$$

Now fix $g \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $|g| \le f_1 + f_2$. For $i = 1, 2$, let

$$g_i := \begin{cases} \dfrac{f_i g}{f_1 + f_2}, & \text{if } f_1 + f_2 > 0 \\ 0, & \text{if } f_1 + f_2 = 0. \end{cases}$$

Then $g_1, g_2 \in C_c(\mathbb{R}^n, \mathbb{R}^m)$, $g = g_1 + g_2$ and

$$|g_i| \leq \frac{f_i}{f_1 + f_2} |g| \leq f_i.$$

Taking supremum in g we have

$$\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2). \tag{3.6}$$

Inequalities (3.5) and (3.6) now give us the claim. $\square_{\text{Claim 3}}$

Claim 4: For all $f \in C_c^+(\mathbb{R}^n)$,

$$\lambda(f) = \int_{\mathbb{R}^n} f \, \mathrm{d}\mu.$$

Proof of Claim 4. Let $\varepsilon > 0$. Given $f \in C_c^+(\mathbb{R}^n)$ choose $0 = t_0 < t_1 < \ldots < t_N$ such that

$$t_N := 2 \, ||f||_\infty \, ,$$

$0 < t_i - t_{i-1} < \varepsilon$ and $\mu\left(f^{-1}(\{t_i\})\right) = 0$ for all $1 \leq i \leq N$.

This can be done since suppose for instance for all $t \in \left(\frac{\varepsilon}{2}, \varepsilon\right)$, $\mu\left(f^{-1}(\{t\})\right) > 0$. Now

$$\begin{aligned} \infty > \mu\left(\operatorname{spt} f\right) &\geq \mu\left(f^{-1}\left(\left(\frac{\varepsilon}{2}, \varepsilon\right)\right)\right) \\ &= \mu\left(\cup_{t \in \left(\frac{\varepsilon}{2}, \varepsilon\right)} f^{-1}(\{t\})\right). \end{aligned}$$

For $i \in \mathbb{N}$, define

$$T_i := \left\{ t \in \left(\frac{\varepsilon}{2}, \varepsilon\right) : \frac{1}{i+1} \leq \mu\left(f^{-1}(\{t\})\right) < \frac{1}{i} \right\}.$$

Then

$$\begin{aligned} \mu\left(\bigcup_{t \in \left(\frac{\varepsilon}{2}, \varepsilon\right)} f^{-1}(\{t\}) \right) &\geq \mu\left(\bigcup_{i=1}^{\infty} \left(\bigcup_{t \in T_i} f^{-1}(\{t\}) \right) \right) \\ &= \sum_{i=1}^{\infty} \mu\left(\bigcup_{t \in T_i} f^{-1}\left(\{t\}\right) \right). \end{aligned}$$

There exists T_j with $\operatorname{card} T_j = \infty$. Pick sequence $(t_k)_{k=1}^{\infty} \subset T_j$.

$$\begin{aligned} \sum_{i=1}^{\infty} \mu\left(\cup_{t \in T_i} f^{-1}(\{t\})\right) &\geq \mu\left(\cup_{t \in T_j} f^{-1}(\{t\})\right) \geq \mu\left(\cup_{k=1}^{\infty} f^{-1}(\{t_k\})\right) \\ &= \sum_{k=1}^{\infty} \mu\left(f^{-1}(\{t_k\})\right) \geq \frac{1}{j+1} \sum_{k=1}^{\infty} 1 = \infty. \end{aligned}$$

This is a contradiction. Same for any open interval.

So define

$$U_j := f^{-1}((t_{j-1}, t_j)).$$

Then U_j is open, $\mu(U_j) < \infty$. By Theorem 1.16, there exist compact sets K_j such that $K_j \subset U_j$ and $\mu(U_j - K_j) < \frac{\varepsilon}{N}$ for all $j \in \{1, \ldots, N\}$. Also there exist functions $g_j \in C_c(\mathbb{R}^n, \mathbb{R}^m)$, with $|g_j| \leq 1$, $\operatorname{spt} g_j \subset U_j$ and

$$|L(g_j)| \geq \mu(U_j) - \frac{\varepsilon}{N}.$$

In addition, there exist functions $h_j \in C_c^+(\mathbb{R}^m)$ such that $\operatorname{spt} h_j \subset U_j$, $0 \leq h_j \leq 1$ and $h_j = 1$ on the compact set $K_j \cup \operatorname{spt} g_j$. Then

$$\lambda(h_j) \geq |L(g_j)| \geq \mu(U_j) - \frac{\varepsilon}{N}$$

and

$$\begin{aligned} \lambda(h_j) &= \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq h_j\} \\ &\leq \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq 1, \operatorname{spt} g \subset U_j\} \\ &= \mu(U_j). \end{aligned}$$

Let

$$A := \left\{ x : f(x) \left(1 - \sum_{j=1}^N h_j(x)\right) > 0 \right\}.$$

Since $f(x)\left(1 - \sum_{j=1}^N h_j(x)\right)$ is continuous, A is open.

$$\begin{aligned} \lambda\left(f - f\sum_{j=1}^N h_j\right) &= \sup\left\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq f - f\sum_{j=1}^N h_j\right\} \\ &\leq \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq ||f||_\infty \chi_A\} \\ &= ||f||_\infty \sup\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq \chi_A\} \\ &= ||f||_\infty \mu(A) = ||f||_\infty \mu\left(\cup_{j=1}^N (U_j - \{x : h_j(x) = 1\})\right) \\ &\leq ||f||_\infty \sum_{j=1}^N \mu(U_j - K_j) \leq \varepsilon ||f||_\infty. \end{aligned}$$

Hence

$$\begin{aligned} \lambda(f) &= \lambda\left(f - f\sum_{j=1}^N h_j\right) + \lambda\left(f\sum_{j=1}^N h_j\right) \\ &\leq \varepsilon ||f||_\infty + \sum_{j=1}^N \lambda(fh_j) \leq \varepsilon ||f||_\infty + \sum_{j=1}^N \lambda(t_j h_j) \\ &\leq \varepsilon ||f||_\infty + \sum_{j=1}^N t_j \mu(U_j) \end{aligned}$$

and

$$\begin{aligned}\lambda(f) &\geq \sum_{j=1}^{N} \lambda(fh_j) \geq \sum_{j=1}^{N} t_{j-1}\lambda(h_j) \\ &\geq \sum_{j=1}^{N} t_{j-1}\left(\mu\left(U_j\right) - \frac{\varepsilon}{N}\right) \geq \sum_{j=1}^{N} t_{j-1}\mu\left(U_j\right) - t_N\varepsilon.\end{aligned}$$

Since

$$\sum_{j=1}^{N} t_{j-1}\mu\left(U_j\right) \leq \int_{\mathbb{R}^n} f \, \mathrm{d}\mu \leq \sum_{j=1}^{N} t_j\mu\left(U_j\right)$$

and

$$\sum_{j=1}^{N} t_{j-1}\mu\left(U_j\right) - t_N\varepsilon \leq \lambda(f) \leq \sum_{j=1}^{N} t_j\mu\left(U_j\right) + \varepsilon\left|\left|f\right|\right|_\infty$$

we have

$$\begin{aligned}\left|\lambda(f) - \int_{\mathbb{R}^n} f \, \mathrm{d}\mu\right| &\leq \sum_{j=1}^{N}(t_j - t_{j-1})\mu\left(U_j\right) + t_N\varepsilon + \varepsilon\left|\left|f\right|\right|_\infty \\ &\leq \varepsilon\mu\left(\operatorname{spt} f\right) + 3\varepsilon\left|\left|f\right|\right|_\infty.\end{aligned}$$

Letting $\varepsilon \to 0$ we obtain the claim

$$\lambda(f) = \int_{\mathbb{R}^n} f \, \mathrm{d}\mu.$$

$\square_{\text{Claim 4}}$

Claim 5: There exists a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ satisfying

$$L(f) = \int_{\mathbb{R}^n} f \cdot \sigma \, \mathrm{d}\mu$$

for all $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$.

Proof of Claim 5. For $f \in C_c(\mathbb{R}^n)$ and $e \in \mathbb{R}^n$ such that $|e| = 1$, define

$$\lambda_e(f) := L(fe).$$

Clearly λ_e is linear and

$$\begin{aligned}|\lambda_e(f)| &= |L(fe)| \\ &\leq \sup\left\{|L(g)| : g \in C_c(\mathbb{R}^n, \mathbb{R}^m), |g| \leq |f|\right\} \\ &= \lambda(|f|) = \int_{\mathbb{R}^n} |f| \, \mathrm{d}\mu.\end{aligned}$$

Therefore, by Theorem 3.6, we can extend λ_e to a bounded linear functional on $L^1(\mathbb{R}^n, \mu)$. Since $L^\infty(\mathbb{R}^n, \mu) = L^1(\mathbb{R}^n, \mu)$ (there exists a bijective isometry), by Corollary 3.27, there exists $\sigma_e \in L^\infty(\mathbb{R}^n, \mu)$ such that for all $f \in C_c(\mathbb{R}^n)$

$$\lambda_e(f) = \int_{\mathbb{R}^n} f \sigma_e \, \mathrm{d}\mu.$$

Let $e_1, \ldots, e_m$ be the standard basis for $\mathbb{R}^m$ and define

$$\sigma := \sum_{j=1}^{m} \sigma_{e_j} e_j.$$

Then if $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ we have

$$\begin{aligned} L(f) &= \sum_{j=1}^{m} L((f \cdot e_j)e_j) = \sum_{j=1}^{m} \int (f \cdot e_j)\sigma_{e_j} \, \mathrm{d}\mu \\ &= \int f \cdot \sigma \, \mathrm{d}\mu. \end{aligned}$$

$\square_{\text{Claim 5}}$

Claim 6:

$$|\sigma| = 1 \quad \mu \text{ a.e.}$$

Proof of Claim 6. Let $U \subset \mathbb{R}^n$ be open, $\mu(U) < \infty$. Then

$$\mu(U) = \sup \left\{ \int f \cdot \sigma \, \mathrm{d}\mu : f \in C_c(\mathbb{R}^n, \mathbb{R}^m), |f| \leq 1, \operatorname{spt} f \subset U \right\}.$$

Now take $f_k \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ such that $|f_k| \leq 1$, $\operatorname{spt} f_k \subset U$ and $f_k \cdot \sigma \to |\sigma|$. This can be done because for each k, there exists $f_k \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $\operatorname{spt} f_k \subset U$ such that $\mu(\{x \in U : f_k(x) \neq \tilde{\sigma}(x)\}) < \frac{1}{k}$ where

$$\tilde{\sigma}(x) = \begin{cases} \frac{\sigma(x)}{|\sigma(x)|} = \operatorname{sign} \sigma(x), & \text{if } |\sigma(x)| > 1 \\ 0, & \text{if } |\sigma(x)| = 1. \end{cases}$$

Then $\lim_{k \to \infty} f_k = \tilde{\sigma}$ μ a.e..

(We may assume that for all $k \in \mathbb{N}$

$$\{x \in U : f_k(x) \neq \tilde{\sigma}(x)\} \subset \{x \in U : f_{k-1}(x) \neq \tilde{\sigma}(x)\}.$$

This can be done because $f_k = \sigma|_{K_k}$ for a compact set $K_k \subset U$ and $f_{k+1} = \sigma|_{K_{k+1}}$ for a compact set $K_{k+1} \subset U$. $K_k \cup K_{k+1}$ is also compact and a subset of U, so define

$$\tilde{f}_{k+1} = \sigma|_{K_k \cup K_{k+1}}$$

and substitute f_{k+1}. Then repeat.)

We have

$$\int_U |\sigma| \ \mathrm{d}\mu = \lim_{k\to\infty} \int f_k \cdot \sigma \ \mathrm{d}\mu \le \mu(U).$$

On the other hand, if $f \in C_c(\mathbb{R}^n, \mathbb{R}^m)$ with $|f| \le 1$ and $\operatorname{spt} f \subset U$, then

$$\int f \cdot \sigma \le \int_U |f| \, |\sigma| \ \mathrm{d}\mu \le \int_U |\sigma| \ \mathrm{d}\mu.$$

Taking supremum, we obtain

$$\mu(U) \le \int_U |\sigma| \ \mathrm{d}\mu.$$

So for all open sets $U \subset \mathbb{R}^n$

$$\mu(U) = \int_U |\sigma| \ \mathrm{d}\mu,$$

implying

$$\int_U |\sigma| - 1 \ \mathrm{d}\mu = 0$$

for all open sets $U \subset \mathbb{R}^n$.

Consider sets

$$V_k := \left\{ x \in \mathbb{R}^n : \frac{1}{k} \ge |\sigma| > \frac{1}{k+1} \right\}.$$

For any ε we can find open sets U_k such that $V_k \subset U_k$, $\mu(U_k - V_k) < \varepsilon$. Then

$$\begin{aligned} 0 &= \left| \int_{U_k} |\sigma(x)| - 1 \ \mathrm{d}\mu \right| = \left| \int_{V_k} |\sigma(x)| - 1 \ \mathrm{d}\mu + \int_{U_k - V_k} |\sigma| - 1 \ \mathrm{d}\mu \right| \\ &\ge \frac{\mu(V_k)}{k+1} - \varepsilon(|||\sigma|||_\infty + 1). \end{aligned}$$

Letting $\varepsilon \to 0$ we obtain $\mu(V_k) = 0$. Thus $|\sigma(x)| = 1$ μ a.e. x. $\square_{\text{Claim 6}}$

$\square$

Corollary 3.29. *Assume $L : C_c(R^n) \to \mathbb{R}$ is linear and non-negative in the sense that $L(f) \ge 0$ for all $f \in C_c(\mathbb{R}^n)$ such that $f \ge 0$. Then there exists a Radon measure μ on $\mathbb{R}^n$ such that*

$$L(f) = \int_{\mathbb{R}^n} f \ d\mu$$

for all $f \in C_c^\infty(\mathbb{R}^n)$.

Proof. Choose any compact set $K \subset \mathbb{R}^n$ and select $J \in C_c(\mathbb{R}^n)$ such that $J = 1$ on K and $0 \le J \le 1$. Then for any $f \in C_c(\mathbb{R}^n)$ with $\operatorname{spt} f \subset K$ define

$$g := ||f||_\infty J - f \ge 0.$$

Since L is positive,

$$0 \le L(g) = ||f||_\infty L(J) - L(f),$$

we have

$$L(f) \leq L(J)\, ||f||_\infty \, .$$

By the Riesz Representation Theorem, there exists μ and σ such that

$$L(f) = \int_{\mathbb{R}^n} f\sigma \; \mathrm{d}\mu$$

for all $f \in C_c(\mathbb{R}^n)$ and that $\sigma = \pm 1$ μ a.e..

Suppose there exists set E such that $\mu\,(E) > 0$ and $\sigma(x) = -1$ for all $x \in E$. Then for any $\varepsilon > 0$ there exists compact set $K \subset E$ and open set $U \supset E$ such that

$$\mu\,(E-K) < \varepsilon \quad , \quad \mu\,(U-E) < \varepsilon.$$

Hence choosing ε small enough and $f \in C_c^+(\mathbb{R}^n)$ such that $f = 1$ on K, $\operatorname{spt} f \subset U$, $0 \leq f \leq 1$ (by Urysohn's Lemma), we have $L(f) \geq 0$.

$$\begin{aligned} \int_{\mathbb{R}^n} f\sigma \; \mathrm{d}\mu &= -\int_K f \; \mathrm{d}\mu + \int_{U-K} f\sigma \; \mathrm{d}\mu < -\mu\,(K) + 2\varepsilon \\ &< -\mu\,(U) + 3\varepsilon. \end{aligned}$$

This is a contradiction. Therefore $\sigma = 1$ μ a.e.. □

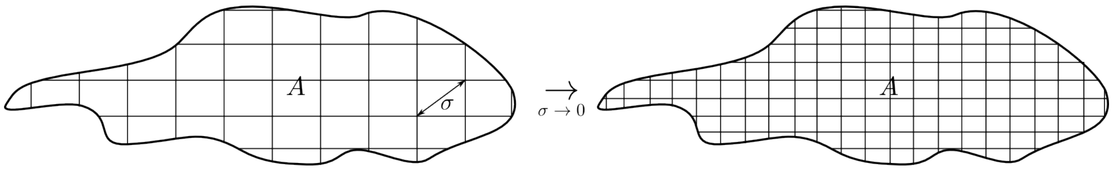

Figure 2: Hausdorff outer measure

4 Hausdorff measure and dimension

Definition 4.1 (Hausdorff outer measure)**.** (i) Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$ and $0 < \sigma \leq \infty$. We define

$$\mathcal{H}^s_\sigma(A) := \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subset \cup_{j=1}^{\infty} C_j, \operatorname{diam} C_j \leq \sigma \right\},$$

where C_j are arbitrary sets,

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma\left(\frac{s}{2}+1\right)}$$

and

$$\Gamma(t) := \int_0^\infty e^{-x} x^{t-1} \, \mathrm{d}x,$$

where $0 < t < \infty$, is the Gamma function.

(ii) For A and s as above, define

$$\mathcal{H}^s(A) := \lim_{\sigma \to 0} \mathcal{H}^s_\sigma(A) = \sup_{\sigma > 0} \mathcal{H}^s_\sigma(A).$$

We call $\mathcal{H}^s$ the *s-dimensional Hausdorff outer measure* on $\mathbb{R}^n$.

Theorem 4.2. *$\mathcal{H}^s$ is a Borel regural outer measure for $0 \leq s < \infty$.*

Proof. **Claim 1:** $\mathcal{H}^s_\sigma$ is an outer measure.

Proof of Claim 1. Let $\{A_k\}_{k=1}^\infty \subset \mathbb{R}^n$ and $A_k \subset \cup_{j=1}^\infty C_j^k$, $\operatorname{diam} C_j^k \leq \sigma$. Then $\cup_{k=1}^\infty A_k \subset \cup_{j,k=1}^\infty C_j^k$ and

$$\mathcal{H}^s_\sigma\left(\cup_{k=1}^\infty A_k\right) \leq \frac{\alpha(s)}{2^s} \sum_{k=1}^\infty \sum_{j=1}^\infty \left(\operatorname{diam} C_j^k\right)^s.$$

Taking infimum for each k, we obtain

$$\mathcal{H}^s_\sigma\left(\cup_{k=1}^\infty A_k\right) \leq \sum_{k=1}^\infty \mathcal{H}^s_\sigma(A_k).$$

$\square_{\text{Claim 1}}$

Claim 2: $\mathcal{H}^s$ is an outer measure.

Proof of Claim 2. Let $\{A_k\}_{k=1}^\infty \subset \mathbb{R}^n$. Then

$$\mathcal{H}^s_\sigma\left(\cup_{k=1}^\infty A_k\right) \le \sum_{k=1}^\infty \mathcal{H}^s_\sigma(A_k) \le \sum_{k=1}^\infty \mathcal{H}^s(A_k).$$

Let $\sigma \to 0$ and we obtain the claim. $\square_{\text{Claim 2}}$

Claim 3: $\mathcal{H}^s$ is a Borel outer measure.

Proof of Claim 3. Let $A, B \subset \mathbb{R}^n$ with $\operatorname{dist}(A,B) > 0$. Pick $0 < \sigma < \frac{1}{4}\operatorname{dist}(A,B)$. Suppose $A \cup B \subset \cup_{k=1}^\infty C_k$ and $\operatorname{diam} C_k \le \sigma$. Let

$$\begin{aligned} \mathcal{A} &:= \{C_j : C_j \cap A \ne \emptyset\}, \\ \mathcal{B} &:= \{C_j : C_j \cap B \ne \emptyset\}. \end{aligned}$$

Then $A \subset \cup_{C_j \in \mathcal{A}} C_j$, $B \subset \cup_{C_j \in \mathcal{B}} C_j$ and $C_i \cap C_j = \emptyset$ if $C_i \in \mathcal{A}$, $C_j \in \mathcal{B}$. Hence

$$\begin{aligned} \frac{\alpha(s)}{2^s}\sum_{j=1}^\infty (\operatorname{diam} C_j)^s &\ge \frac{\alpha(s)}{2^s}\sum_{C_j\in\mathcal{A}} (\operatorname{diam} C_j)^s + \frac{\alpha(s)}{2^s}\sum_{C_j\in\mathcal{B}} (\operatorname{diam} C_j)^s \\ &\ge \mathcal{H}^s_\sigma(A) + \mathcal{H}^s_\sigma(B). \end{aligned}$$

Taking the infimum over all such sets $\{C_j\}_{j=1}^\infty$ we have

$$\mathcal{H}^s_\sigma(A \cup B) \ge \mathcal{H}^s_\sigma(A) + \mathcal{H}^s_\sigma(B)$$

for all $0 < \sigma < \frac{1}{4}\operatorname{dist}(A,B)$. Let $\sigma \to 0$ and we obtain

$$\mathcal{H}^s(A \cup B) \ge \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

So by Carathéodory's Criterion we have that $\mathcal{H}^s$ is Borel. $\square_{\text{Claim 3}}$

Claim 4: $\mathcal{H}^s$ is a Borel regular outer measure.

Proof of Claim 4. Note that $\operatorname{diam} \overline{C} = \operatorname{diam} C$ for all $C \subset \mathbb{R}^n$. So

$$\mathcal{H}^s_\sigma(A) = \frac{\alpha(s)}{2^s}\inf\left\{\sum_{j=1}^\infty (\operatorname{diam} C_j)^s : A \subset \cup_{j=1}^\infty C_j, \operatorname{diam} C_j \le \sigma, C_j \text{ closed}\right\}.$$

Let A be an arbitrary subset of $\mathbb{R}^n$ such that $\mathcal{H}^s_\sigma(A) < \infty$ for all $\sigma > 0$. For each $k \ge 1$, choose closed sets $\{C_j^k\}_{j=1}^\infty$ so that $\operatorname{diam} C_j^k \le \frac{1}{k}$, $A \subset \cup_{j=1}^\infty C_j^k$ and

$$\frac{\alpha(s)}{2^s}\sum_{j=1}^\infty \left(\operatorname{diam} C_j^k\right)^s \le \mathcal{H}^s_{\frac{1}{k}}(A) + \frac{1}{k}.$$

Let

$$A_k := \cup_{j=1}^\infty C_j^k \quad , \quad B := \cap_{k=1}^\infty A_k.$$

$\square$

Then B is Borel. Also $A \subset A_k$ for all $k \in \mathbb{N}$ so $A \subset B$. Furthermore

$$\begin{aligned} \mathcal{H}^s_{\frac{1}{k}}(B) &\le \frac{\alpha(s)}{2^s}\sum_{j=1}^{\infty}(\operatorname{diam} C_j^k)^s \\ &\le \mathcal{H}^s_{\frac{1}{k}}(A) + \frac{1}{k} \end{aligned}$$

for all $k \in \mathbb{N}$. Letting $k \to \infty$, we obtain

$$\mathcal{H}^s(B) \le \mathcal{H}^s(A).$$

Also $A \subset B$, so

$$\mathcal{H}^s(A) \le \mathcal{H}^s(B).$$

Thus

$$\mathcal{H}^s(B) = \mathcal{H}^s(A).$$

Suppose that there exists $\sigma > 0$ such that $\mathcal{H}^s_\sigma(A) = \infty$. Then for any closed $\{C_j\}_{j=1}^{\infty}$ with $\operatorname{diam} C_j \le \sigma$ and such that $A \subset \cup_{j=1}^{\infty} C_j$ we have that the set

$$B := \cup_{j=1}^{\infty} C_j$$

is Borel and

$$\mathcal{H}^s_\sigma(B) \ge \mathcal{H}^s_\sigma(A) = \infty.$$

Therefore

$$\mathcal{H}^s(A) = \mathcal{H}^s(B) = \infty.$$

□

Theorem 4.3 (Elementary Properties of Hausdorff Outer Measure)**.**

(i) $\mathcal{H}^0$ *is counting measure.*

(ii) $\mathcal{H}^1 = \mathcal{L}^1$ *on* $\mathbb{R}$.

(iii) $\mathcal{H}^s = 0$ *on* $\mathbb{R}^n$ *for all* $s > n$.

(iv) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}(A)$ *for all* $\lambda > 0$, $A \subset \mathbb{R}^n$.

(v) $\mathcal{H}^s(L(A)) = \mathcal{H}^s(A)$ *for* $A \subset \mathbb{R}^n$ *and for each affine isometry*

$$L : \mathbb{R}^n \to \mathbb{R}^n.$$

Proof. (iv) and (v) follow straight from the definition.

For part (i): $s = 0$, $\alpha(0) = \frac{1}{\Gamma(1)} = 1$, $\{a\} \subset \{a\}$, $\operatorname{diam}\{a\} = 0$. Then for any σ, $\mathcal{H}^0_\sigma(\{a\})$, so $\mathcal{H}^0(\{a\}) = 1$.

For part (ii): Let $A \subset \mathbb{R}$, $\sigma > 0$. Then

$$\begin{aligned}
\mathcal{L}^1(A) &= \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam} C_j : A \subset \cup_{j=1}^{\infty} C_j\right\} \\
&\leq \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam} C_j : A \subset \cup_{j=1}^{\infty} C_j, \operatorname{diam} C_j \leq \sigma\right\} \\
&= \mathcal{H}^1_\sigma(A),
\end{aligned}$$

since $\frac{\alpha(1)}{2} = \frac{\sqrt{\pi}}{2\Gamma(\frac{2}{3})} = 1$.

Given any $\sigma > 0$, let for all $k \in \mathbb{Z}$,

$$I_k := [k\sigma, (k+1)\sigma].$$

Then $\operatorname{diam}(C_j \cap I_k) \leq \operatorname{diam} C_j$ and $\sum_{k\in\mathbb{Z}} \operatorname{diam}(C_j \cap I_k) \leq \operatorname{diam} C_j$. Hence

$$\begin{aligned}
\mathcal{L}^1(A) &= \inf\left\{\sum_{j=1}^{\infty} \operatorname{diam} C_j : A \subset \cup_{j=1}^{\infty} C_j\right\} \\
&\geq \inf\left\{\sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \operatorname{diam}(C_j \cap I_k) : A \subset \cup_{j=1}^{\infty} C_j\right\} \\
&\geq \mathcal{H}^1_\sigma(A).
\end{aligned}$$

So $\mathcal{L}^1(A) \geq \mathcal{H}^1(A)$.

$\therefore \mathcal{L}^1(A) = \mathcal{H}^1(A)$ for all $A \subset \mathbb{R}^n$.

For part (iii): Fix $m \in \mathbb{N}$ and let Q be the unit cube in $\mathbb{R}^n$. Q can be decomposed into m^n cubes of sidelength $\frac{1}{m}$ and diameter $\frac{\sqrt{2}}{m}$. Then

$$\mathcal{H}^s_{\frac{\sqrt{n}}{m}}(Q) = 0$$

if $s > n$. Therefore $\mathcal{H}^s(Q) = 0$ and $\mathcal{H}^s(\mathbb{R}^n) = 0$. □

Theorem 4.4. *Suppose $A \subset \mathbb{R}^n$ and $\mathcal{H}^s_\sigma(A) = 0$ for some $0 < \sigma \leq \infty$. Then $\mathcal{H}^s(A) = 0$.*

Proof. Let $s = 0$. If $\mathcal{H}^s_\sigma(A) = 0$ for some $0 < \sigma \leq \infty$ then $A = \emptyset$, implying $\mathcal{H}^s(A) = 0$.

Let $s > 0$. Fix $\varepsilon > 0$. There exists sets $\{C_j\}_{j=1}^{\infty}$ such that $A \subset \cup_{j=1}^{\infty} C_j$ and $\frac{\alpha(s)}{2^s} \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s \leq \varepsilon$. Therefore for each $j \in \mathbb{N}$,

$$\operatorname{diam} C_j \leq 2\left(\frac{\varepsilon}{\alpha(s)}\right)^{\frac{1}{s}} =: \sigma(\varepsilon).$$

Hence $\mathcal{H}^s_{\sigma(\varepsilon)}(A) \leq \varepsilon$. Letting $\varepsilon \to 0$, it follows that $\sigma(\varepsilon) \to 0$, implying $\mathcal{H}^s(A) = 0$. □

Theorem 4.5. *Let $A \subset \mathbb{R}^n$ and $0 \leq s < t < \infty$.*

(i) If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.

(ii) If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = \infty$.

Proof. Proof of part (i). Let $\mathcal{H}^s(A) < \infty$ and $\sigma > 0$. Then there exist sets $\{C_j\}_{j=1}^{\infty}$ such that $\operatorname{diam} C_j \leq \sigma$, $A \subset \cup_{j=1}^{\infty} C_j$ and $\frac{\alpha(s)}{2^s} \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s \leq \mathcal{H}_\sigma^s(A) + 1$. Then

$$\begin{aligned}
\mathcal{H}_\sigma^s(A) &\leq \frac{\alpha(t)}{2^t} \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^t \\
&= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \frac{\alpha(s)}{2^s} \sum_{j=1}^{\infty} (\operatorname{diam} C_j)^s (\operatorname{diam} C_j)^{t-s} \\
&\leq \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sigma^{t-s} \left(\mathcal{H}^s(A) + 1\right).
\end{aligned}$$

Letting $\sigma \to 0$, we have that $\mathcal{H}^t(A) = 0$, proving part (i). $\square_{\text{(i)}}$

Proof of part (ii). Proof of part (ii) follows from the proof of of part (i). $\square_{\text{(ii)}}$

$\square$

Definition 4.6 (Hausdorff dimension)**.** The *Hausdorff dimension* of a set $A \subset \mathbb{R}^n$ is defined to be

$$\mathcal{H}_{\dim} := \inf \{s \in [0, \infty) : \mathcal{H}^s(A) = 0\}.$$

Remark 4.7.

- Since $\mathcal{H}^s = 0$ on $\mathbb{R}^n$, when $s > n$, we have $\mathcal{H}_{\dim}(A) \leq n$ for $A \subset \mathbb{R}^n$.
- Let $s := \mathcal{H}_{\dim}(A)$. Then $\mathcal{H}^t(A) = 0$ for all $t > s$ and $\mathcal{H}^t(A) = \infty$ for all $t < s$. Also $\mathcal{H}^s(A) \in [0, \infty]$.

Theorem 4.8. *Let $f : \mathbb{R}^n \to [0, \infty]$ be $\mathcal{L}^n$-measurable. Then the region*

$$A := \{(x, y) : x \in \mathbb{R}^n, y \in \mathbb{R}, 0 \leq y \leq f(x)\}$$

is a $\mathcal{L}^{n+1}$-measurable set.

Proof. Let

$$g(x, y) := f(x) - y$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}$. Then g is a sum of two $\mathcal{L}^{n+1}$-measurable functions on $\mathbb{R}^{n+1}$ and hence it is $\mathcal{L}^{n+1}$-measurable. Therefore

$$A = \{(x, y) : y \geq 0\} \cap \{(x, y) : g(x, y) \geq 0\}$$

is also $\mathcal{L}^{n+1}$-measurable. $\square$

Definition 4.9 (Steiner symmetrisation)**.** Let $a, b \in \mathbb{R}^n$, $|a| = 1$. We define the *line through b in the direction a* by

$$L_b^a := \{b + ta : t \in \mathbb{R}\}$$

and the *plane through the origin perpendicular to a* by

$$P_a := \{x \in \mathbb{R}^n : x \cdot a = 0\}\,.$$

Furthermore, let $A \subset \mathbb{R}^n$. We define the *Steiner symmetrisation of A with respect to P_a* to be the set

$$S_a(A) := \bigcup_{\substack{b \in P_a \\ A \cap L_b^a = \emptyset}} \left\{ b + ta : |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) \right\}.$$

Theorem 4.10 (Properties of the Steiner Symmetrisation)**.**

(i) $\operatorname{diam} S_a(A) \leq \operatorname{diam} A$

(ii) If A is $\mathcal{L}^n$-measurable, then $S_a(A)$ is also $\mathcal{L}^n$-measurable and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.

Proof. Proof of part (i). Case $\operatorname{diam} A = \infty$ is trivial, so let $\operatorname{diam} A < \infty$. Since $S_a(A) \subset S_a(\overline{A})$, $\operatorname{diam} S_a(A) \leq \operatorname{diam} S_a(\overline{A})$ and $\operatorname{diam} A = \operatorname{diam} \overline{A}$, so we may assume that A is closed.

Given any $\varepsilon > 0$, select $x, y \in S_a(A)$ such that

$$\operatorname{diam} S_a(A) \leq |x - y| + \varepsilon.$$

Let

$$b := x - (x \cdot a)a \quad , \quad c := y - (y \cdot a)a.$$

be the projections of x and y onto P_a.
Let also

$$\begin{aligned} r &:= \inf\{t : b + ta \in A\} \\ s &:= \sup\{t : b + ta \in A\} \\ u &:= \inf\{t : c + ta \in A\} \\ v &:= \sup\{t : c + ta \in A\}\,. \end{aligned}$$

Without a loss of generality we may assume that $u - r \geq s - u$ (otherwise change x and y pairwise). Then

$$\begin{aligned} v - r &\geq \frac{1}{2}(v - r) + \frac{1}{2}(s - u) = \frac{1}{2}(s -) + \frac{1}{2}(v - u) \\ &\leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

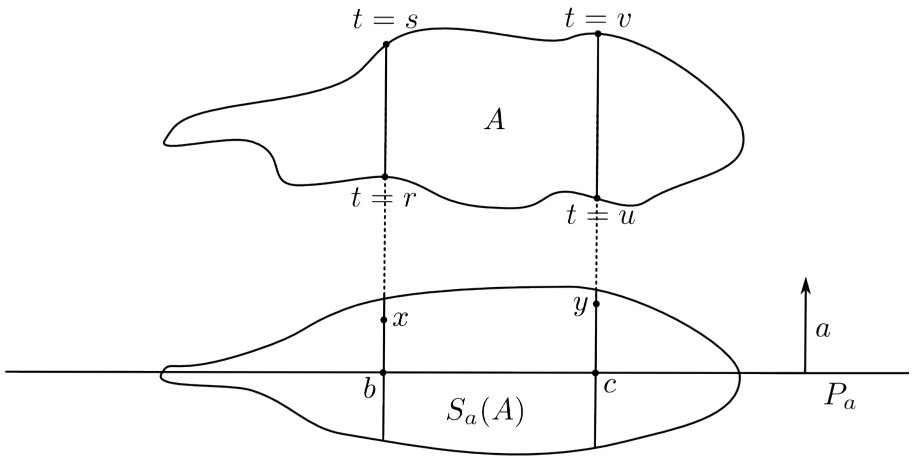

Figure 3: Proof of Theorem 4.10

and

$$\begin{aligned} |x \cdot a| &\leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) \\ |y \cdot a| &\leq \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Thus,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|\,,$$

implying

$$\begin{aligned} (\operatorname{diam} S_a(A) - \varepsilon)^2 &\leq |x-y|^2 \\ &= |b-c|^2 + |x \cdot a - y \cdot a|^2 \ \text{(Pythagoras)} \\ &\leq |b-c|^2 + |v-r|^2 \\ &= |(b+ra) - (c+va)|^2 \ \text{(Pythagoras)} \\ &\leq (\operatorname{diam} A)^2\,, \end{aligned}$$

since A is closed and so $b + ra, c + va \in A$. Letting $\varepsilon \to 0$ we obtain

$$\operatorname{diam} S_a(A) \leq \operatorname{diam} A.$$

$\square_{\text{(i)}}$

Proof of part (ii). We may assume that $a = e_n = (0, \ldots, 0, 1)$ which means $P_a = \mathbb{R}^{n-1}$. Since $\mathcal{H}^1 = \mathcal{L}^1$ on $\mathbb{R}$, the map $f : \mathbb{R}^{n-1} \to \mathbb{R}$ defined by

$$f(b) := \mathcal{H}^1(A \cap L_b^a)$$

is $\mathcal{L}^{n-1}$-measurable by Fubini's Theorem and

$$\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b)\ \mathrm{d}b.$$

Then

$$S_a(A) = \left\{(b,y) : b \in \mathbb{R}^{n-1}, \frac{-f(b)}{2} \le y \le \frac{f(b)}{2}\right\} - \left\{(b,0) : b \in \mathbb{R}^{n-1} L_b^a \cap A = \emptyset\right\}$$

is $\mathcal{L}^{n-1}$-measurable by Theorem 4.8 and because $\mathcal{L}^n\left(\{(b,0) : L_b^a \cap A = \emptyset\}\right) = 0$.

We also have that

$$\mathcal{L}^n\left(S_a(A)\right) = \int_{\mathbb{R}^{n-1}} f(b)\ \mathrm{d}b = \mathcal{L}^n(A)$$

by Fubini's Theorem. $\square_{\text{(ii)}}$

$\square$

Theorem 4.11 (Isodiametric Inequality)**.** *For all sets $A \subset \mathbb{R}^n$,*

$$\mathcal{L}^n(A) \le \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n.$$

Proof. If $\operatorname{diam} A = \infty$, there is nothing to prove. Suppose $\operatorname{diam} A < \infty$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Define

$$\begin{aligned} A_1 &:= S_{e_1}(A), \\ A_2 &:= S_{e_2}(A_1), \\ &\vdots \\ A_n &:= S_{e_n}(A_{n-1}). \end{aligned}$$

For this process we use notation

$$A^* := A_n.$$

Claim 1: A^* is symmetric with respect to the origin.

Proof of Claim 1. By induction. A_1 is symmetric with respect to P_{e_1}. Let $1 \le k < n$ and suppose that A_k is symmetric w.r.t $P_{e_1}, \ldots, P_{e_k}$. Clearly A_{k+1} is symmetric w.r.t $P_{e_{k+1}}$.

Fix $1 \le j \le k$ and let $S_j : \mathbb{R}^n \to \mathbb{R}$ be the reflection through P_{e_j}. Let $b \in P_{e_{k+1}}$. Since $S_j(A_k) = A_k$ for all $1 \le j \le k$,

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}})$$

implying

$$\{t : b + te_{k+1} \in A_{k+1}\} = \{t : S_j b + te_{k+1} \in A_{k+1}\},$$

thus $S_j(A) = A_{k+1}$.

By induction $A^* = A_n$ is symmetric w.r.t $P_{e_1}, \ldots, P_{e_n}$ and so symmetric w.r.t the origin. $\square_{\text{Claim 1}}$

Claim 2: $\mathcal{L}^n(A^*) \le \alpha(n)\left(\frac{\operatorname{diam} A^*}{2}\right)^n$.

Proof of Claim 2. Let $x \in A^*$. Then $-x \in A^*$ and so $\operatorname{diam} A^* \geq 2|x|$. Therefore $A^* \subset B\left(0, \frac{\operatorname{diam} A^*}{2}\right)$ and

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{\operatorname{diam} A^*}{2}\right)\right) = \alpha(n)\left(\frac{\operatorname{diam} A^*}{2}\right)^n.$$

$\square_{\text{Claim 2}}$

Claim 3: $\mathcal{L}^n(A) \leq \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n$.

Proof of Claim 3. $\overline{A}$ is $\mathcal{L}^n$-measurable so by Theorem 4.10 $\mathcal{L}^n\left((\overline{A})^*\right) = \mathcal{L}^n(\overline{A})$ and $\operatorname{diam}(\overline{A})^* \leq \operatorname{diam} \overline{A}$. Therefore,

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\overline{A}) = \mathcal{L}\left((\overline{A})^*\right) \leq \alpha(n)\left(\frac{\operatorname{diam}(\overline{A})^*}{2}\right)^n \\ &\leq \alpha(n)\left(\frac{\operatorname{diam} \overline{A}}{2}\right)^2 = \alpha(n)\left(\frac{\operatorname{diam} A}{2}\right)^n. \end{aligned}$$

$\square_{\text{Claim 3}}$

$\square$

Theorem 4.12. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.*

Proof. **Claim 1:** $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$ for all $A \subset \mathbb{R}^n$.

Proof of Claim 1. Fix $\sigma > 0$. Let $\{C_j\}_{j=1}^\infty$ be such that $A \subset \cup_{j=1}^\infty C_j$ and $\operatorname{diam} C_j \leq \sigma$. By the Isodiametric Inequality

$$\mathcal{L}^n(A) \leq \sum_{j=1}^\infty \mathcal{L}^n(C_j) \leq \sum_{j=1}^\infty \alpha(n)\left(\frac{\operatorname{diam} C_j}{2}\right)^n.$$

Taking infimum on the right hand side we have $\mathcal{L}^n(A) \leq \mathcal{H}^n_\sigma(A)$ implying $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$. $\square_{\text{Claim 1}}$

From the definition of $\mathcal{L}^n = \mathcal{L}^1 \times \ldots \times \mathcal{L}^1$ we have that for all $A \subset \mathbb{R}^n$ and $\sigma > 0$,

$$\mathcal{L}^n(A) \inf\left\{\sum_{i=1}^\infty \mathcal{L}^n(Q_i) : Q_i \text{ cubes}, A \subset \cup_{i=1}^\infty Q_i, \operatorname{diam} Q_i \leq \sigma\right\}$$

where the cubes have sides parallel to the coordinate axels.

Claim 2: $\mathcal{H}^n$ is absolutely continuous with respect to $\mathcal{L}^n$.

Proof of Claim 2. For each cube $Q \subset \mathbb{R}^n$

$$\alpha(n)\left(\frac{\operatorname{diam} Q}{2}\right)^n = \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n \mathcal{L}^n(Q).$$

Therefore

$$\begin{aligned}
\mathcal{H}^n_\sigma &\le \inf\left\{\sum_{i=1}^\infty \alpha(n)\left(\frac{\operatorname{diam} Q}{2}\right)^n : Q_i \text{ cubes}, A\subset \cup_{i=1}^\infty Q_i, \operatorname{diam} Q_i \le \sigma\right\} \\
&= \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n \mathcal{L}^n(A).
\end{aligned}$$

Thus

$$\mathcal{H}^n(A) = \lim_{\sigma\to 0}\mathcal{H}^n_\sigma(A) \le \alpha(n)\left(\frac{\sqrt{n}}{2}\right)^n \mathcal{L}^n(A).$$

$\square_{\text{Claim 2}}$

Claim 3: $\mathcal{H}^n(A) \le \mathcal{L}^n(A)$ for all $A\subset\mathbb{R}^n$.

Proof of Claim 3. Fix $\sigma > 0$, $\varepsilon > 0$. We can find cubes $\{Q\}_{j=1}^\infty$ such that $A \subset \cup_{i=1}^\infty Q_i$, $\operatorname{diam} Q_i \le \sigma$ and

$$\sum_{i=1}^\infty \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \varepsilon.$$

By Corollary 2.4, for each i there exist disjoint closed balls $\{B^i_k\}_{k=1}^\infty$ contained in $\operatorname{int} Q_i$ such that $\operatorname{diam} B^i_k \le \sigma$,

$$\mathcal{L}^n(Q_i - \cup_{k=1}^\infty B^i_k) = \mathcal{L}^n(\operatorname{int} Q_i - \cup_{k=1}^\infty B^i_k) = 0.$$

By Claim 2,

$$\mathcal{H}^n(Q_i - \cup_{k=1}^\infty B^i_k) = 0.$$

Therefore

$$\begin{aligned}
\mathcal{H}^n_\sigma(A) &\le \sum_{i=1}^\infty \mathcal{H}^n_\sigma(Q_i) = \sum_{i=1}^\infty \mathcal{H}^n_\sigma(\cup_{k=1}^\infty B^i_k) \\
&\le \sum_{i=1}^\infty\sum_{k=1}^\infty \mathcal{H}^n_\sigma(B^i_k) \le \sum_{i=1}^\infty\sum_{k=1}^\infty \alpha(n)\left(\frac{\operatorname{diam} B^i_k}{2}\right)^n \\
&= \sum_{i=1}^\infty\sum_{k=1}^\infty \mathcal{L}^n(B^i_k) = \sum_{i=1}^\infty \mathcal{L}^n(Q_i) \\
&\le \mathcal{L}^n(A) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ and $\sigma \to 0$ we obtain the claim. $\square_{\text{Claim 3}}$

$\square$

Theorem 4.13. *Suppose that $E\subset\mathbb{R}^n$, E is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(E) < \infty$. Then*

$$\lim_{r\to 0}\frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} = 0$$

for $\mathcal{H}^s$-a.e. $x \in E^C$.

Proof. Fix $t > 0$ and let

$$A_t := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

Since E is $\mathcal{H}^s$-measurable with $\mathcal{H}^s(E) < \infty$, $\mathcal{H}^s|_E$ is a Radon outer measure and so for each $\varepsilon > 0$ there exists a compact set $K \subset E$ such that

$$\mathcal{H}^s(E - K) \leq \varepsilon.$$

Let

$$U := \mathbb{R}^n - K.$$

Then U is open and $A_t \subset E^C \subset U$. Fix $\delta > 0$ and consider family

$$\mathcal{F} := \left\{ B(x,r) : B(x,r) \subset U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

$\mathcal{F}$ covers A_t. By the Vitali Covering Theorem, there exists a countable, disjoint family of balls $\{B_i\}_{i=1}^{\infty}$ in $\mathcal{F}$ such that $A_t \subset \cup_{i=1}^{\infty} \tilde{B}_i$, where $\tilde{B}_i$ is concentric with B_i and has 5 times the radius of B.

Write $B_i = B(x_i, r_i)$. Then

$$\begin{aligned} \mathcal{H}^s_{10\delta}(A_t) &\leq \sum_{i=1}^{\infty} \alpha(s)(5r_i)^s \leq \frac{5^s}{t} \sum_{i=1}^{\infty} \mathcal{H}^s(B_i \cap E) \\ &\leq \frac{5^s}{t} \mathcal{H}^s(U \cap E) = \frac{5^s}{t} \mathcal{H}^s(E - K) \leq \frac{5^s \varepsilon}{t}. \end{aligned}$$

Letting $\delta \to 0$ we have $\mathcal{H}^s(A_t) \leq \frac{5^s \varepsilon}{t}$. Since ε was arbitrary we have $\mathcal{H}(A_t) = 0$ for all $t > 0$. □

Theorem 4.14. *Assume $E \subset \mathbb{R}^n$, E is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(E) < \infty$. Then*

$$\frac{1}{2^s} \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \leq 1$$

for $\mathcal{H}^s$-a.e. $x \in E$.

Proof. **Claim 1:** $\limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} \leq 1$ for $\mathcal{H}^s$-a.e. $x \in E$.

Proof of Claim 1. Fix $t > 1$ and define

$$A_t := \left\{ x \in E : \limsup_{r \to 0} \frac{\mathcal{H}^s(B(x,r) \cap E)}{\alpha(s) r^s} > t \right\}.$$

Fix $\varepsilon > 0$. Since $\mathcal{H}^s|_E$ is Radon, there exists open set U such that $U \supset A_t$ with

$$\mathcal{H}^s(U \cap E) \leq \mathcal{H}^s(A_t) + \varepsilon.$$

Define

$$\mathcal{F} := \left\{ B(x,r) : B(x,r) \subset U, 0 < r < \delta, \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} > t \right\}.$$

By Corollary 2.3, since $\mathcal{F}$ is a fine cover of A_t, there exists a countable disjoint family of balls $\{B_i\}_{i=1}^\infty$ in $\mathcal{F}$ such that

$$A_t \subset (\cup_{i=1}^\infty B_i) \cup \left(\cup_{i=m+1}^\infty \tilde{B}_i\right)$$

for each $m \in \mathbb{N}$. Write $B_i = B(x_i, r_i)$. Then

$$\begin{aligned}
\mathcal{H}^s_{10\delta} &\leq \sum_{i=1}^m \alpha(s) r_i^s + \sum_{i=m+1}^\infty \alpha(s)(5r_i)^s \\
&\leq \frac{1}{t}\sum_{i=1}^m \mathcal{H}^s(B_i\cap E) + \frac{5^s}{t}\sum_{i=m+1}^\infty \mathcal{H}^s(B_i\cap E) \\
&\leq \frac{1}{t}\mathcal{H}^s(U\cap E) + \frac{5^s}{t}\mathcal{H}^s\left(\cup_{i=m+1}^\infty B_i \cap E\right).
\end{aligned}$$

Letting $m \to \infty$

$$\begin{aligned}
\lim_{m\to\infty} \mathcal{H}^s\left(\cup_{i=m+1}^\infty B_i\cap E\right) &= \mathcal{H}^s\left(\cap_{m=1}^\infty \left(\cup_{i=m+1}^\infty B_i\right)\cap E\right) \\
&= \mathcal{H}^s(\emptyset) = 0
\end{aligned}$$

because the B_i are disjoint so if $x \in \cup_{i=2}^\infty B_i$, there exists j such that $x \in B_j$ and $x \notin B_i$ for $i \neq j$ and so $x \notin \cup_{i=j+1}^\infty B_i$. So $\mathcal{H}^s_{10\delta}(A_t) \leq \frac{1}{t}\mathcal{H}^s(U\cap E) \leq \frac{1}{t}\left(\mathcal{H}^s(A_t)+\varepsilon\right)$. Letting $\delta \to$ and $\varepsilon \to 0$ we have $\mathcal{H}^s(A_t) \leq \frac{1}{t}\mathcal{H}^s(A_t)$ and since $\mathcal{H}^s(A_t) \leq \mathcal{H}^s(E) < \infty$ we have $\mathcal{H}^s(A_t) = 0$ for all $t > 1$. $\square_{\text{Claim 1}}$

Claim 2: $\limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}$ for $\mathcal{H}^s$-a.e. $x \in E$.

Proof of Claim 2. For $\delta > 0$, $0 < \tau < 1$ we denote by $E(\delta,\tau)$ the set of points $x \in E$ such that

$$\mathcal{H}^s_\delta(C\cap E) \leq \tau\alpha(s)\left(\frac{\operatorname{diam} C}{2}\right)^s$$

whenever $C \subset \mathbb{R}^n$, $x \in C$ and $\operatorname{diam} C \leq \delta$. Then if $\{C_i\}_{i=1}^\infty$ are subsets of $\mathbb{R}^n$ with $\operatorname{diam} C_i \leq \delta$, $E(\delta,\tau) \subset \cup_{i=1}^\infty C_i$, $C_i \cap E(\delta,\tau) \neq \emptyset$ for all i, we have

$$\begin{aligned}
\mathcal{H}^s_\delta(E(\delta,\tau)) &\leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(C_i\cap E(\delta,\tau)) \\
&\leq \sum_{i=1}^\infty \mathcal{H}^s_\delta(C_i\cap E) \\
&\leq \tau\sum_{i=1}^\infty \alpha(s)\left(\frac{\operatorname{diam} C}{2}\right)^s.
\end{aligned}$$

Hence, taking infimum over covers $\{C_i\}_{i=1}^{\infty}$ with $\operatorname{diam} C_i \leq \delta$ we have $\mathcal{H}^s_\delta(E(\delta,\tau)) \leq \mathcal{H}^s_\delta(E) < \infty$ and $0 < \tau < 1$ and therefore $\mathcal{H}^s_\delta(E(\delta,\tau)) = 0$. In particular $\mathcal{H}^s_\delta(E(\delta, 1-\delta)) = 0$ and so by Theorem 4.4,

$$\mathcal{H}^s(E(\delta, 1-\delta)) = 0.$$

Now if $x \in E$ and

$$\limsup_{r\to 0} \frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} < \frac{1}{2^s}$$

then there exists $0 < \delta < 1$ such that

$$\limsup_{r\to 0} \frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} \leq \frac{1-\delta}{2^s}$$

for all $0 < r \leq \delta$. So if $x \in C$ (any C) and $\operatorname{diam} C \leq \delta$,

$$\begin{aligned}
\mathcal{H}^s_\delta(C\cap E) &= \mathcal{H}^s_\infty(C\cap E) \text{ (since } \operatorname{diam}(C\cap E)\leq\delta) \\
&\leq \mathcal{H}^s_\infty\left(B(x,\operatorname{diam} C)\cap E\right) \\
&\leq (1-\delta)\alpha(s)\left(\frac{\operatorname{diam} C}{2}\right)^s.
\end{aligned}$$

So $x \in E(\delta, 1-\delta)$ and therefore

$$\left\{x\in E : \limsup_{r\to 0} \frac{\mathcal{H}^s_\infty(B(x,r)\cap E)}{\alpha(s)r^s} < \frac{1}{2^s}\right\} \subset \cup_{k=1}^{\infty} E(\frac{1}{k}, 1-\frac{1}{k}).$$

Since

$$\begin{aligned}
\mathcal{H}^s(B(x,r)\cap E) &= \sup_{0<\delta\leq\infty} \mathcal{H}^s_\delta(B(x,r)\cap E) \\
&\geq \mathcal{H}^s_\infty(B(x,r)\cap E)
\end{aligned}$$

we have

$$\limsup_{r\to 0} \frac{\mathcal{H}^s(B(x,r)\cap E)}{\alpha(s)r^s} \geq \frac{1}{2^s}$$

for $\mathcal{H}^s$-a.e. $x \in E$. $\square_{\text{Claim 2}}$

$\square$

Definition 4.15 (Lipschitz function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called *Lipschitz*, if there exists a constant C such that

$$|f(x) - f(y)| \leq C\,|x-y|$$

for all $x, y \in \mathbb{R}^n$. We define

$$\operatorname{Lip}(f) := \sup\left\{\frac{|f(x)-f(y)|}{|x-y|} : x, y\in\mathbb{R}^n, x\neq y\right\}.$$

Theorem 4.16. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $A \subset \mathbb{R}^n$, $0 \le s < \infty$. Then*

$$\mathcal{H}^s(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s(A).$$

Proof. Let $\delta > 0$ and $A \subset \cup_{i=1}^{\infty} C_i$, $\operatorname{diam} C_i \le \delta$. Then $\operatorname{diam} f(C_i) \le \mathrm{Lip}(f)\delta$ and

$$f(A) \subset \cup_{i=1}^{\infty} f(C_i).$$

Then

$$\mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(A)) \le \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} f(C_i)}{2}\right)^s \le (\mathrm{Lip}(f))^s \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\operatorname{diam} f(C_i)}{2}\right)^s.$$

Taking infimum over such C_i we have

$$\mathcal{H}^s_{\mathrm{Lip}(f)\delta}(f(A)) \le (\mathrm{Lip}(f))^s \mathcal{H}^s_\delta(A)$$

and letting $\delta \to 0$ we are done. □

Definition 4.17 (Graph of a function)**.** For $f : \mathbb{R}^n \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$ we define

$$G(f; A) := \{(x, f(x) : x \in A\} \subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}.$$

We call $G(f; A)$ the *graph of f over A.*

Theorem 4.18. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$, $A \subset \mathbb{R}^n$, $\mathcal{L}^n(A) > 0$. Then*

(i) $\mathcal{H}_{dim}(G(f; A)) \ge n$ and

(ii) if f is Lipschitz, $\mathcal{H}_{dim}(G(f; A)) = n$.

Proof. Proof of part (i). Let $P : \mathbb{R}^{n+m} \to \mathbb{R}^n$ be the projection to the first n coordinates. Then P is Lipschitz with $\mathrm{Lip}(P) = 1$, hence by Theorem 4.16

$$\mathcal{H}^n(G(f; A)) \ge \mathcal{H}^n(A) > 0,$$

implying $\mathcal{H}_{\mathrm{dim}}(G(f; A)) \ge n$. $\square_{\text{(i)}}$

Proof of part (ii). Let Q denote any cube in $\mathbb{R}^n$ of side length 1. Subdivide Q into k^n subcubes of sidelength $\frac{1}{k}$. Call these cubes $Q_1, \ldots, Q_{k^n}$. Then

$$\operatorname{diam} Q_i = \frac{\sqrt{n}}{k}.$$

Let

$$a^i_j := \min_{x \in Q_j} f^i(x)\ ,\ b^i_j := \max_{x \in Q_j} f^i(x),$$

where $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, k^n\}$ and $f(x) = (f^1(x), \ldots, f^m(x))$.

Since f is Lipschitz,

$$\left|b^i_j - a^i_j\right| \le \mathrm{Lip}(f) \operatorname{diam} Q_j = \mathrm{Lip}(f)\frac{\sqrt{n}}{k}.$$

Now let

$$C_j := Q_j \times \prod_{i=1}^{m} [a_j^i, b_j^i].$$

Then $\{(x, f(x)) : x \in Q_j \cap A\} \subset C_j$ and $\operatorname{diam} C_j \leq \frac{C}{k}$, where C is a constant. Since $G(f; A \cap Q) \subset \cup_{j=1}^{k} C_j$. We have

$$\begin{aligned} \mathcal{H}^n_{\frac{c}{k}}(G(f; A \cap Q)) &\leq \sum_{j=1}^{k^n} \alpha(n) \left(\frac{\operatorname{diam} C_j}{2} \right)^n \\ &\leq k^n \alpha(n) \left(\frac{C}{2k} \right)^n = \alpha(n) \left(\frac{C}{2} \right)^n . \end{aligned}$$

Letting $k \to \infty$, we have $\mathcal{H}^n(G(f; A \cap Q)) \leq \alpha(n) \left(\frac{c}{2}\right)^n$, implying $\mathcal{H}_{\text{dim}}(G(f; A \cap Q)) \leq n$.

Since this is valid for all cubes of sidelength 1, we have $\mathcal{H}_{\text{dim}}(G(f; A)) \leq n$, implying

$$\mathcal{H}_{\text{dim}}(G(f; A)) = n.$$

$\square_{\text{(ii)}}$

$\square$

Theorem 4.19. *Let $f \in L^1_{loc}(\mathbb{R}^n, \mathcal{L}^n)$. Suppose $0 \leq s < n$ and let*

$$A_s := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f| \; dy > 0 \right\}.$$

Then $\mathcal{H}^s(A_s) = 0$.

Proof. Assume first that $f \in L^1(\mathbb{R}^n, \mathcal{L}^n)$. By the Lebesgue-Besicovitch Differentiation Theorem

$$\lim_{r \to 0} \frac{1}{\mathcal{L}^n(B(x,r))} \int_{B(x,r)} |f| \; \mathrm{d}y = |f|$$

$\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. Therefore $\lim_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f| \; \mathrm{d}y = 0$ $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. Since $0 \leq s < n$, $\mathcal{L}^n(A_s) = 0$.

Now fix $\varepsilon > 0$, $\delta > 0$, $\sigma > 0$. Since $f \in L^1(\mathbb{R}^n, \mathcal{L}^n)$, there exists U and $\eta > 0$ such that $\mathcal{L}^n(U) \leq \eta$ and that $\int_U |f| \; \mathrm{d}x < \sigma$. (Continuity of Lebesgue measure).

Let

$$A_s^{\varepsilon} := \left\{ x \in \mathbb{R}^n : \limsup_{r \to 0} \frac{1}{r^s} \int_{B(x,r)} |f| \; \mathrm{d}y > \varepsilon \right\}.$$

Then $\mathcal{L}^n(A_s^{\varepsilon}) \leq \mathcal{L}^n(A_s) = 0$. Thus there exists an open set U such that $A_s^{\varepsilon} \subset U$ and $\mathcal{L}^n(U) < \eta$.

Define

$$\mathcal{F} := \left\{ B(x,r) : x \in A_s^{\varepsilon}, 0 < r < \sigma, B(x,r) \subset U, \frac{1}{r^s} \int_{B(x,r)} |f| \; \mathrm{d}y > \varepsilon \right\}.$$

By the Vitali Covering Theorem, there exist disjoint balls $\{B_i\}_{i=1}^{\infty} \subset \mathcal{F}$ such that $A_s^{\varepsilon} \subset \cup_{i=1}^{\infty} \tilde{B}_i$.

Denoting the radius of B_i by r_i, we have that

$$\begin{aligned} \mathcal{H}^s_{10\delta}(A_s^{\varepsilon}) &\leq \sum_{i=1}^{\infty} \alpha(s)\,(r_i)^s = \frac{\alpha(s)5^s}{\varepsilon} \sum_{i=1}^{\infty} \int_{B_i} |f| \, \mathrm{d}y \\ &\leq \frac{\alpha(s)5^s}{\varepsilon} \int_U |f| \, \mathrm{d}y \leq \frac{\alpha(s)5^s}{\varepsilon} \sigma. \end{aligned}$$

Letting $\delta \to 0$ we have $\mathcal{H}^s(A_s^{\varepsilon}) \leq \frac{\alpha(s)5^s}{\varepsilon}\sigma$.

Letting $\sigma \to 0$ we have $\mathcal{H}^s(A_s^{\varepsilon}) = 0$ for all ε. Thus

$$\mathcal{H}^s(A_s) = 0.$$

For general $f \in L^1_{\text{loc}}(\mathbb{R}^n, \mathcal{L}^n)$, we have

$$A_s \subseteq \cup_Q \left\{ x \in \operatorname{int}(2Q) : \limsup_{r\to 0} \frac{1}{r^s} \int_{B(x,r)} |f| \, \chi_{2Q} \, \mathrm{d}y > 0 \right\},$$

where we have decomposed $\mathbb{R}^n = \cup Q$ where Q are cubes of sidelength 1. Then

$$\mathcal{H}^s(A_s) \leq \sum_Q \mathcal{H}^s(A_s^Q) = 0.$$

$\square$

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